Signals and Systems – Part II: Systems

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1. Systems

2. Types of systems

3. LTI Systems

4. LTI Filter Design in the Frequency Domain

General definition of a system

- A system may be thought of as a **Black Box** (B.B.) with one or more input terminals and one or more output terminals.
- This Black Box could be:
  - ▶ a mechanical system
  - ▶ an electrical system
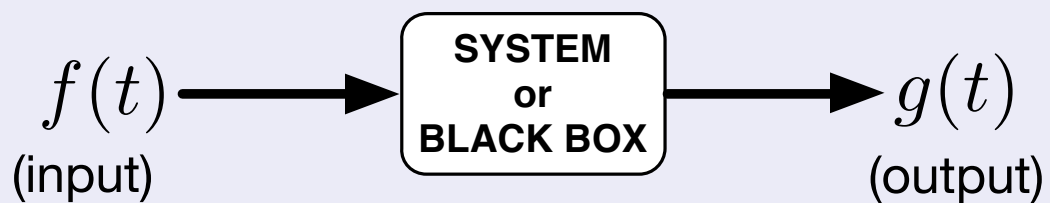 - ▶ a chemical system
  - ▶ a biological system
  - ▶ . . .

  or could be an imaging, or an audio device/process.

How do we describe systems or B.B.?

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Types of systems

System or Black Box

$$f(t) \longrightarrow \boxed{\begin{array}{c}\textbf{SYSTEM}\\ \textbf{or}\\ \textbf{BLACK BOX}\end{array}} \longrightarrow g(t)$$

(input)   (output)

Most Common Systems Types

A	Linear	Nonlinear
B	Time Invariant	Time Variant
C	Causal	Acausal
D	Open Loop	Closed Loop

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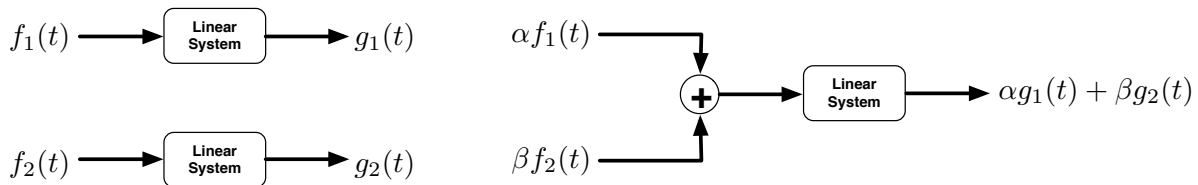
A. Linear vs Nonlinear Systems

For a linear system, the following **must** hold:

IF $f_1(t)$ (input) $\overset{System}{\Rightarrow} g_1(t)$ (output)

AND $f_2(t)$ (input) $\overset{System}{\Rightarrow} g_2(t)$ (output)

THEN $\boxed{\alpha f_1(t) + \beta f_2(t) \overset{System}{\underset{Linear}{\Rightarrow}} \alpha g_1(t) + \beta g_2(t), \quad \forall f_1(t) \neq 0 \text{ and } f_2(t) \neq 0}$

In other words, for a linear system, if the input is a linear combination of inputs, the output will be the same linear combination of the outputs corresponding to these inputs.

B. Time Invariant vs Time Variant Systems

For time invariant systems, the mapping between the input and the output does not depend on the time at which the input signal starts. Mathematically time invariance amounts to the following property:

IF $f(t)$ (input) $\overset{System}{\Rightarrow} g(t)$ (output)

THEN $\boxed{f(t-\tau) \overset{System}{\underset{Time\ Invariant}{\Rightarrow}} g(t-\tau), \quad \forall \tau \in \mathbb{R}}$

In other words, for a time invariant system, if the input signal is time-shifted by τ then the output signal will be shifted by the same time-shift τ.

C. Causal vs Acausal Systems

A system is said to be *causal* if the output of the system, $g(t)$, is only dependent on the values of the input to the system, $f(t)$, for times up until the current point in time, t.

$$f(t) \longrightarrow \boxed{\begin{array}{c} \textbf{SYSTEM} \\ \textbf{or} \\ \textbf{BLACK BOX} \end{array}} \longrightarrow g(t)$$
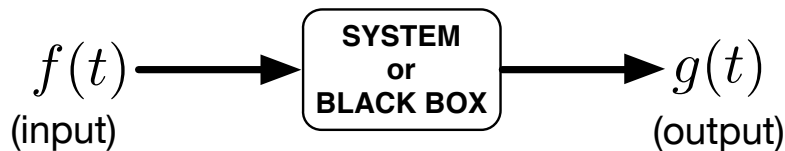
(input)   (output)

For all t, $g(t)$ can only depend on $f(t - \tau)$ for values of $\tau \geq 0$. In other words, for causal systems, τ can never take negative values.

D. Open Loop vs Closed Loop Systems

In a closed loop system (some proportion of) the output signal is fed back to the input signal.
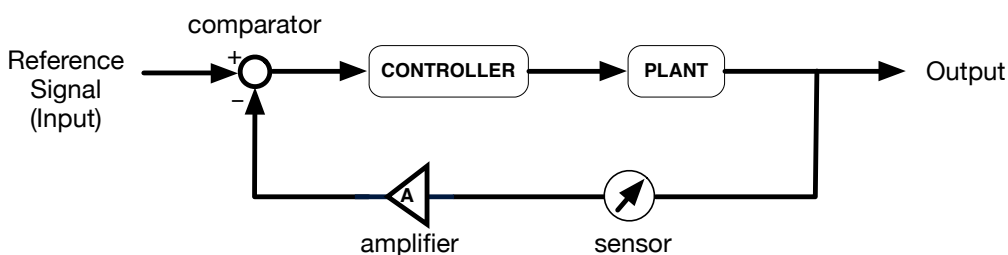
Examples of closed-loop systems include:

- Thermostats for automatic temperature regulation around a desired reference temperature, irrespective of temperature perturbations (e.g. caused by open window(s) or door(s)).
- (Adaptive) Cruise Control for automatic speed regulation in cars (or automatic distance regulation with respect to other vehicles on the road), irrespective of the slope of the road or number of passengers.
- Escalator speed regulation for maintaining constant speed, irrespective of the number of people on the escalator.
- Microphone & amplifier in feedback and the "Larsen" effect: The amplification level is critical to trigger/avoid the "Larsen" effect.

LTI Systems

Definition of an LTI System

An LTI system is a system that is both linear and time invariant.

Why are LTI systems important?

LTI systems are important because any LTI system can be *completely characterised* by a "signal" known as its *impulse response*.

Definition of the impulse response of a system

The impulse response of a system is the output of the system obtained in response to a δ-"function" ("impulse") at its input.
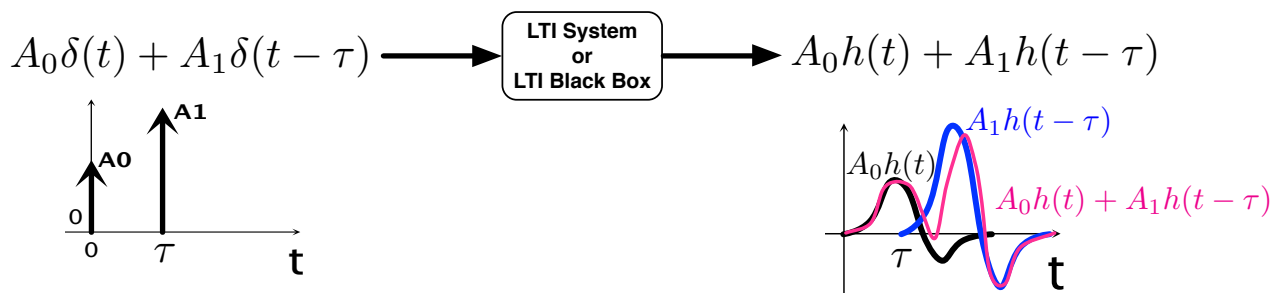
LTI Systems and the Impulse Response

As a consequence of the combination of linearity and time invariance, if the input of an LTI system is:

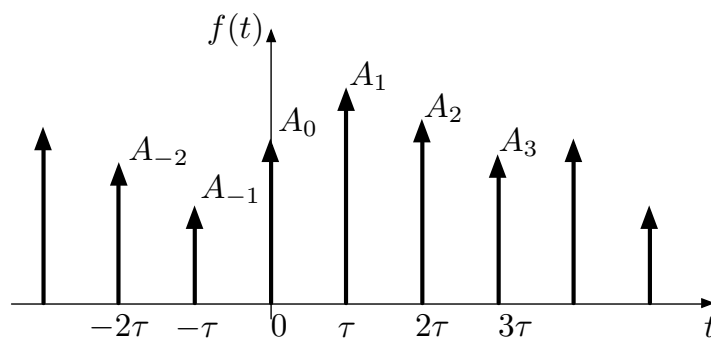$$f(t) = A_0\delta(t) + A_1\delta(t - \tau)$$

then the output of this LTI system will be:

$$g(t) = A_0 h(t) + A_1 h(t - \tau)$$

$A_0\delta(t) + A_1\delta(t - \tau) \longrightarrow$ | LTI System or LTI Black Box | $\longrightarrow A_0 h(t) + A_1 h(t - \tau)$

What happens if there are more δ-"functions" at the input?

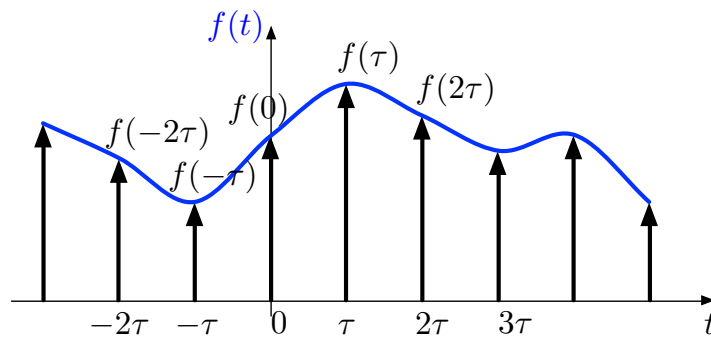# Link between the input and the output of an LTI system

If the input of an LTI system has the following definition:

$$f(t) = \sum_{n=-\infty}^{\infty} A_n\delta(t - n\tau)$$

Then the output of that LTI system will be

$$g(t) = \sum_{n=-\infty}^{+\infty} A_n h(t - n\tau)$$

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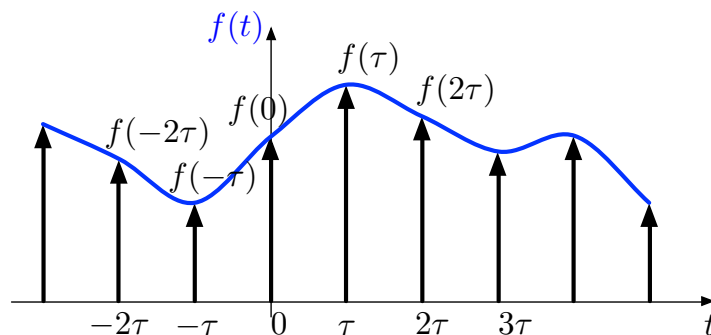
LTI Systems and the Impulse Response

Therefore, $f(t)$ and $g(t)$ can be expressed as:

$$f(t) = \sum_{-\infty}^{+\infty} f(n\tau)\delta(t - n\tau)$$

and

$$g(t) = \sum_{-\infty}^{+\infty} f(n\tau)h(t - n\tau).$$

LTI Systems and the Impulse Response

As $\tau \to 0$, the summations become integrations,
and we thus obtain:

$$
\begin{aligned}
f(t) &= \int_{-\infty}^{+\infty} f(\tau)\delta(t - \tau)d\tau \quad (n\tau \text{ gets replaced by } \tau) \\
&= f(t) * \delta(t) \quad \text{(by the sifting property of the } \delta\text{-"function")}
\end{aligned}
$$

and, similarly:

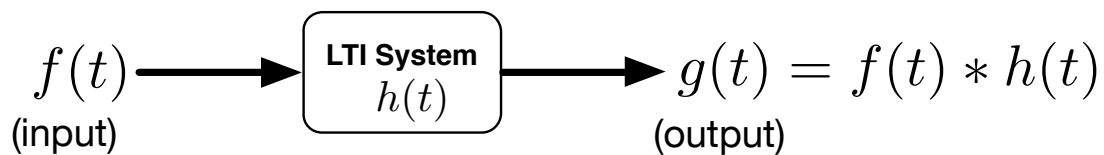$$g(t) = \int_{-\infty}^{+\infty} f(\tau)h(t - \tau)d\tau = f(t) * h(t)$$

LTI Systems and the Impulse Response

Output of an LTI system in terms of its impulse response

For any LTI system, the output $g(t)$ can always be expressed as the *convolution* of the input $f(t)$ with the impulse response of this LTI system $h(t)$:

$$\boxed{g(t) = f(t) * h(t) = h(t) * f(t)} \tag{1}$$

$$f(t) \xrightarrow{\hspace{2cm}} \boxed{\begin{array}{c}\text{LTI System}\\ h(t)\end{array}} \xrightarrow{\hspace{2cm}} g(t) = f(t) * h(t)$$

(input)   (output)

Transfer Function: the Fourier Transform of the Impulse Response

We can also look at the expression (1) in the frequency domain by taking the Fourier transform of both sides:

$$
\begin{aligned}
FT\{g(t)\} &= FT\{f(t) * h(t)\} \\
&= FT\{f(t)\}FT\{h(t)\}
\end{aligned}
$$

which implies:

Output of an LTI system in terms of its transfer function

$$\boxed{G(j\omega) = F(j\omega)H(j\omega) = H(j\omega)F(j\omega)} \tag{2}$$

where $H(j\omega) = FT\{h(t)\}$ is the *transfer function* of the LTI system.

Summary

The **impulse response** or the **transfer function** of a Linear Time Invariant (LTI) system *each* **completely characterise** the input-output properties of that system.

Given the input to an LTI system, the output can be determined:

- **In the time domain:** as the **convolution** of the impulse response and the input.
- **In the frequency domain:** as the **multiplication** of the transfer function and the Fourier transform of the input.
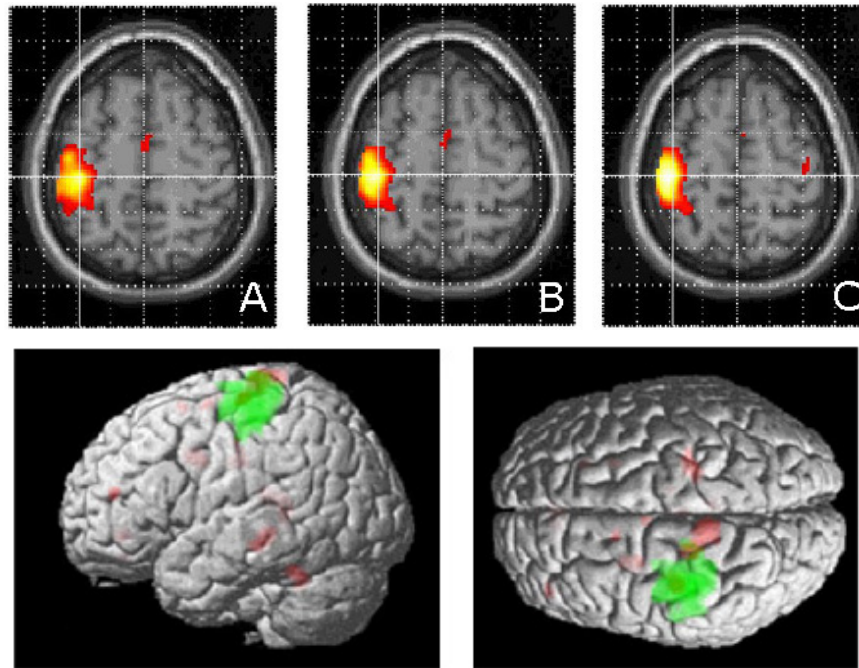
They are **related** as follows: The transfer function is the Fourier transform of the impulse response.

Summary
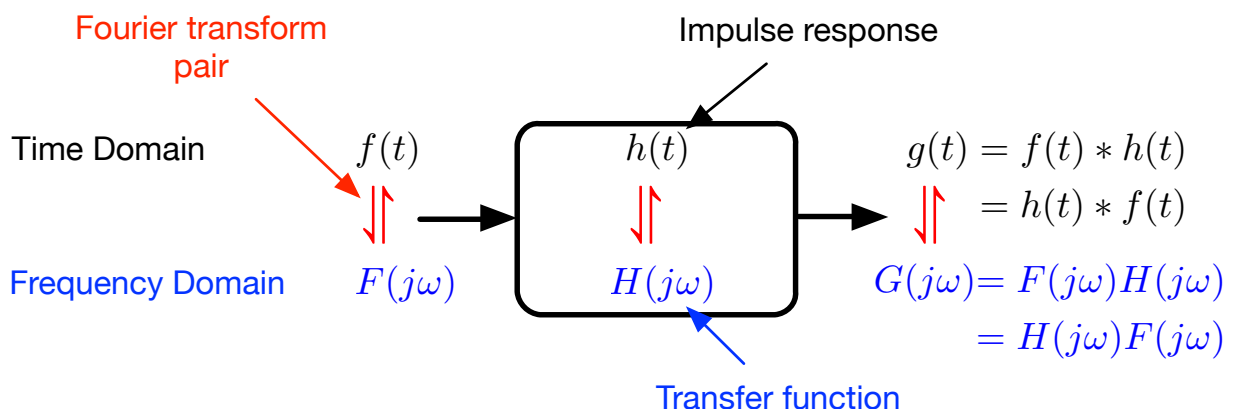
# Examples of the use of Impulse Responses in industry

- Search for task-specific functional regions within the brain using fMRI scans

- 3D Audio Virtual Reality Systems (Matlab Demo)

# Response of LTI systems to sinusoidal inputs

If a pure sinusoid is input into an (asymptotically stable) LTI system, then the output will also settle down, eventually, to a pure sinusoid. This *steady-state* output will have the same frequency as the input but will have a different magnitude and phase. The dependence of the magnitude and phase on the frequency of the input is called the *frequency response* of the system.

$$f(t) = \sin(\omega t)$$

$$g(t) = |H(j\omega)| \sin\left(\omega t + \angle H(j\omega)\right)$$
$$+ \text{ starting transient}$$
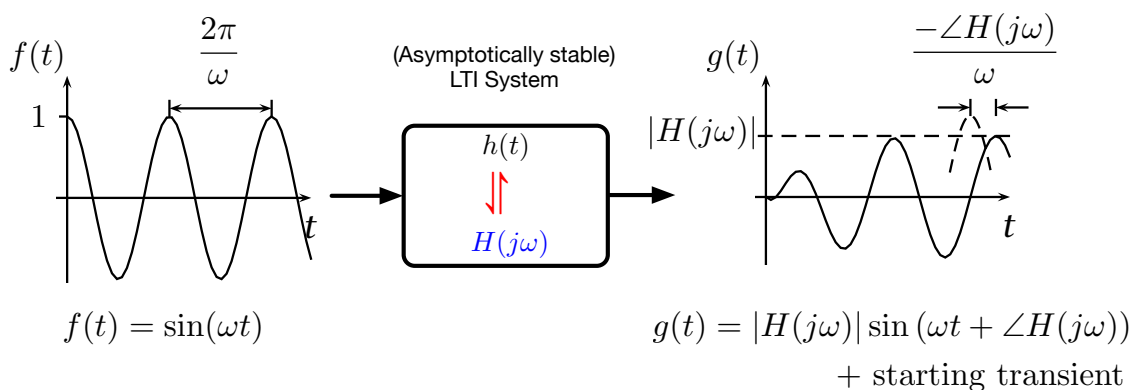
Bode magnitude and phase diagrams

The frequency response can be captured through *Bode* diagrams, which consist in two separate graphs:

- one of $\mathbf{20\log_{10}}|\mathbf{H(j\omega)}|$ (in decibels, dB, i.e. $20\log_{10}$ axis) **vs** ω (in rad/s, $\log_{10}$ axis), i.e. the Bode *magnitude* diagram.[1]
- one of $\angle\mathbf{H(j\omega)}$ (in degrees or radians, linear axis) **vs** ω (in rad/s, $\log_{10}$ axis), i.e. the Bode *phase* diagram.
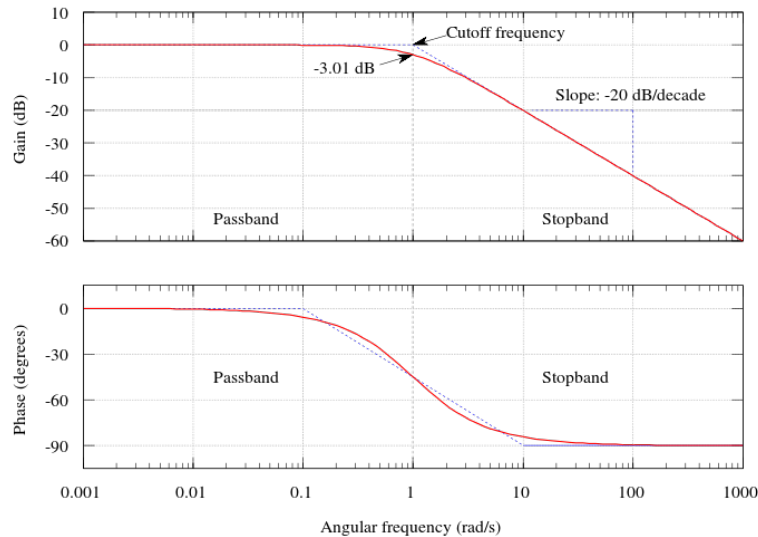
[1]In some books, the Bode *magnitude* diagram is called the Bode *gain* diagram.

1. Systems

2. Types of systems

3. LTI Systems

4. LTI Filter Design in the Frequency Domain

LTI Filter Design in the Frequency Domain

Linear filtering represents a large application class for LTI systems. In what follows, we will emphasise the duality between time-domain and frequency-domain representations of signals (Bode diagrams) and their use for the design and realisation of basic LTI filters. We will, in particular, focus on:

- Broad-spectrum signals: Pseudo-random noise (e.g. Maximum Length Sequences) vs Sweep signals (e.g. Sine Sweeps) vs impulse.
- Low-Pass, High-Pass, Band-Pass, and Band-Stop Filters and (examples of) their corresponding transfer functions.
- Butterworth Filter: the general form of the Butterworth filter can be used as one way of specifying a transfer function and therefore an impulse response.
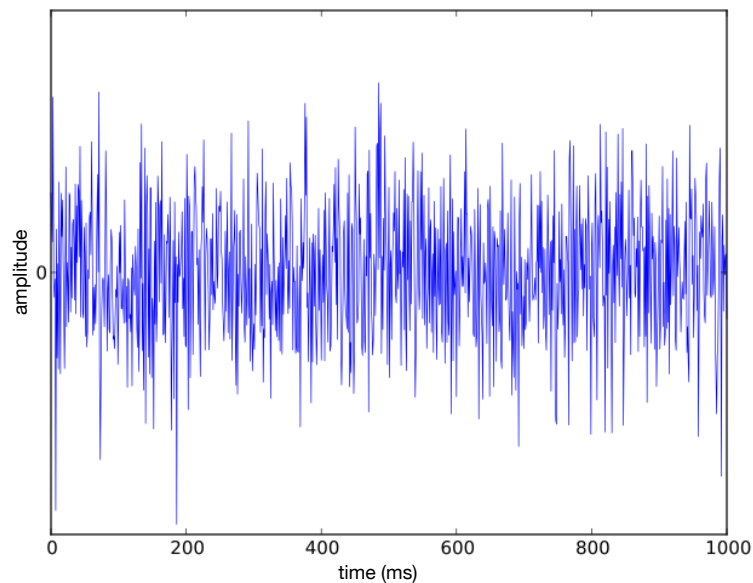
Impulse Bode Diagrams

Exercise

What do you think the Bode magnitude and phase diagrams for an impulse signal (δ-"function") would look like? Hint: Consider the Fourier transform of the δ-"function". What does that tell you about the Bode magnitude and phase of a δ-"function"?
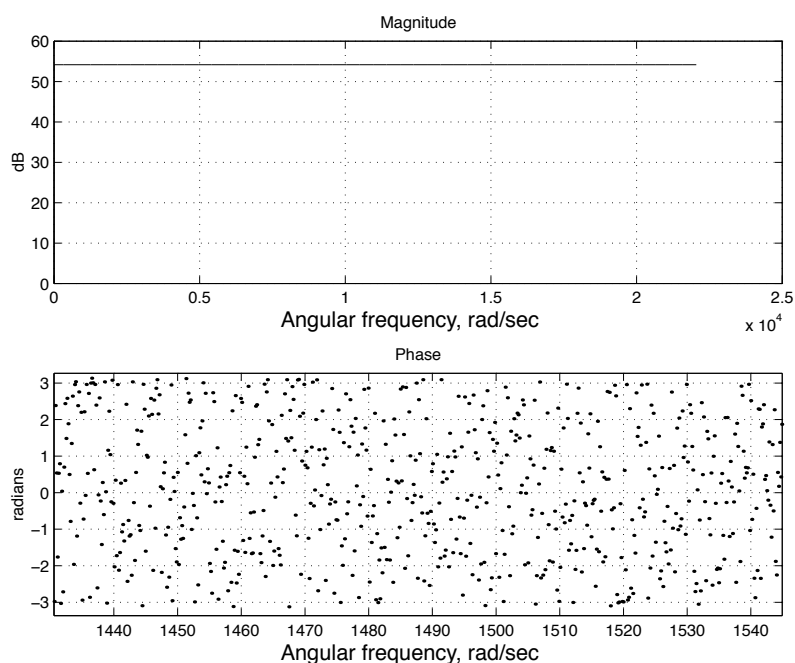
Pseudo-random noise signals

A **Maximum Length Sequence** (MLS) signal of any desired length can be easily generated to approximate a white noise signal:
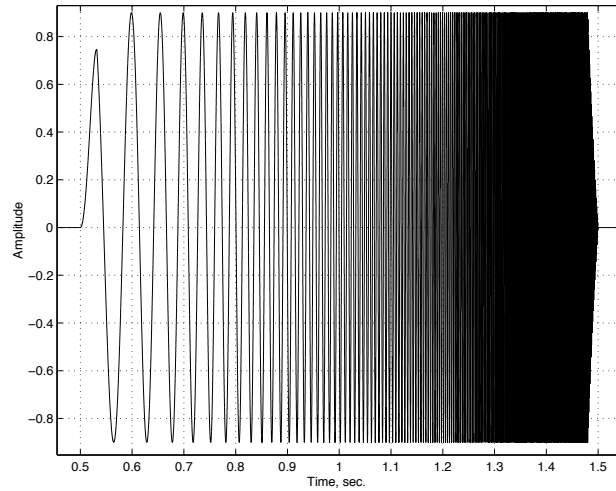
MLS Bode Diagrams

The Bode magnitude and phase diagrams of an MLS signal look like this:

Sine Sweep or Chirp Signals

A **Sine Sweep (or Chirp)** signal is another example of a broad-spectrum signal:
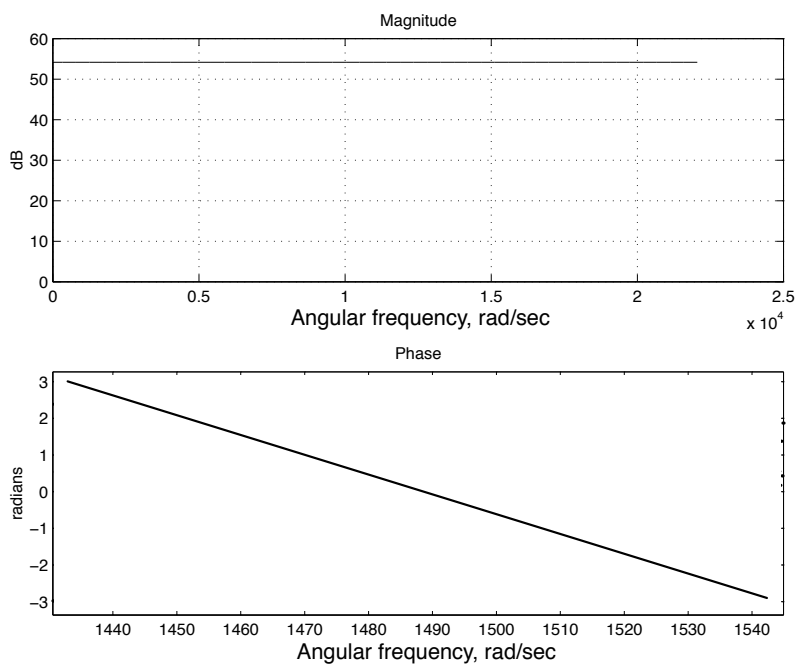

> ### Exercise
> *What do you think the Bode magnitude and phase diagrams for a linear sine sweep would look like?*

Sine Sweep or Chirp Bode Diagrams

The Bode magnitude and phase diagrams of a linear sine sweep signal look like this:

Bode Magnitude Diagrams for Basic Filters

There are five major types of LTI filters. Hereafter, we provide a characterisation of the first 4 in terms of their Bode magnitude diagrams.

Low Pass filter

High pass filter

Band pass filter

Band reject filter
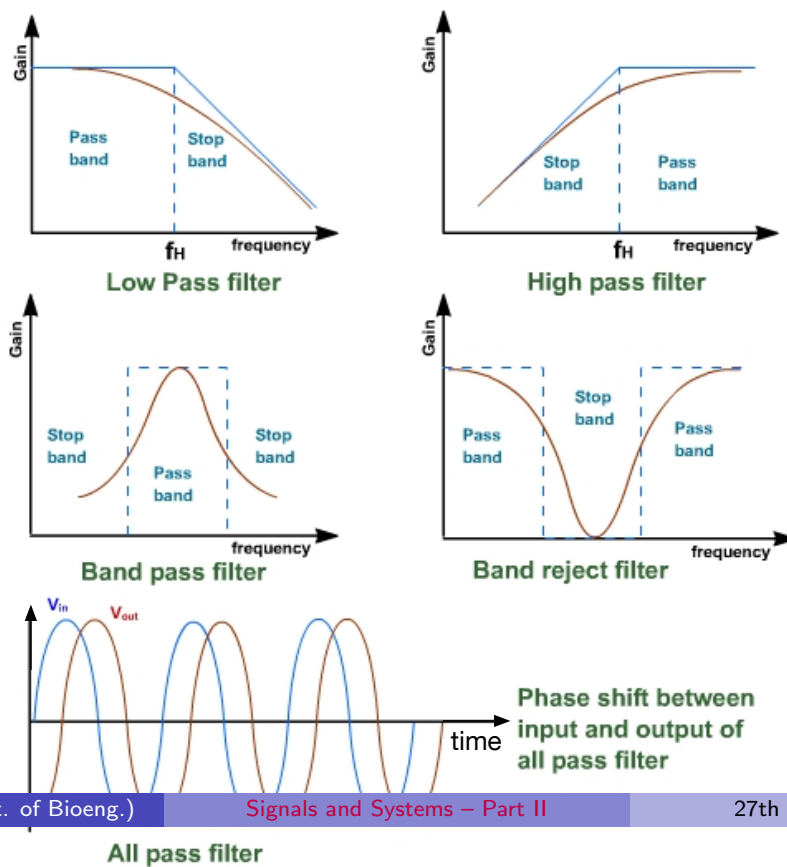
All pass filter

Examples of Transfer Functions for Basic Filters

- First Order Low-Pass Filter Transfer Function: $K\frac{1}{1+\tau j\omega}$ where $\omega_c = \frac{1}{\tau}$ is the cutoff angular frequency of the Low-Pass filter.
- First Order High-Pass Filter Transfer Function: $K\frac{\tau j\omega}{1+\tau j\omega}$ where $\omega_c = \frac{1}{\tau}$ is the cutoff angular frequency of the High-Pass filter.
- Band-Pass = cascade of High-Pass and Low-Pass filters where the cutoff angular frequency of the High-Pass is smaller than the cutoff angular frequency of the Low-Pass.
- Band-Stop = parallel combination of Low-Pass and High-Pass filters where the cutoff angular frequency of the Low-Pass is smaller than the cutoff angular frequency of the High-Pass.

Exercise

Plot the Bode diagrams of the above three filters (low-pass, high-pass, and band-pass) for values of the parameters (K and cutoff angular frequencies) that you chose yourself.

Sketching Bode diagrams

Basic idea: Consider a transfer function written as a ratio of factorised polynomials, e.g.

$$H(j\omega) = \frac{a_1(j\omega)a_2(j\omega)}{b_1(j\omega)b_2(j\omega)}$$

Clearly:

$$\log_{10}|H(j\omega)| = \log_{10}|a_1(j\omega)| + \log_{10}|a_2(j\omega)| - \log_{10}|b_1(j\omega)| - \log_{10}|b_2(j\omega)|,$$

so we can compute the Bode magnitude curve by simply adding and subtracting magnitudes corresponding to terms in the numerator and denominator. Similarly:

$$\angle H(j\omega) = \angle a_1(j\omega) + \angle a_2(j\omega) - \angle b_1(j\omega) - \angle b_2(j\omega)$$
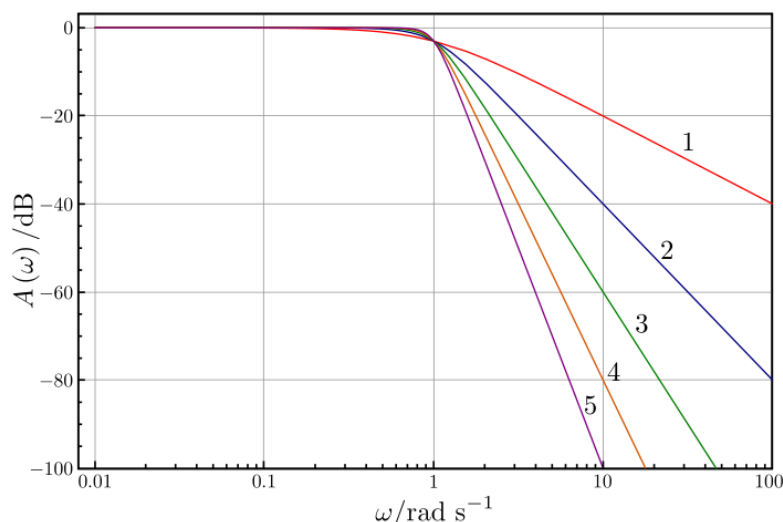
and so the Bode phase curve can be determined in an analogous fashion.

The Butterworth filter: Bode magnitude specification

Consider the following *Butterworth filter* Bode magnitude specification:

$$|H(j\omega)|^2 = \frac{1}{1 + (\omega/\omega_c)^{2N}} \tag{3}$$

where N determines the order of the filter and ω_c determines the cutoff angular frequency.

The Butterworth filter: Bode magnitude specification

Let us consider the case of a Butterworth filter of order $N = 1$ with a cutoff angular frequency $\omega_c = 1$ rad/s, so that

$$|H(j\omega)|^2 = \frac{1}{1 + \omega^2}$$

Because this is only a specification on the Bode *magnitude* of the filter, one still has to decide on what the Bode *phase* specification will be.

> **Remark**
>
> *There are some standard options for specifying the phase, which depend on how the filter will be implemented (e.g. analogue implementation or digital implementation). For the time being, we will just look at a "standard" analogue filter implementation. This imposes the analytical expression of the (continuous-time) transfer function and thereby the Bode phase plot. For an analogue implementation of this Butterworth filter, we will here consider the transfer function:* $H(j\omega) = \frac{1}{1 + j\omega}$.

Butterworth filter: Bode magnitude and phase plots

An example of the magnitude and Bode phase plots for an analogue Butterworth filter of order $N = 1$ with cutoff freqency $\omega_c = 1$ rad/s is provided hereafter. As you can see the Bode diagrams are exactly those that we considered for first order low-pass filters.
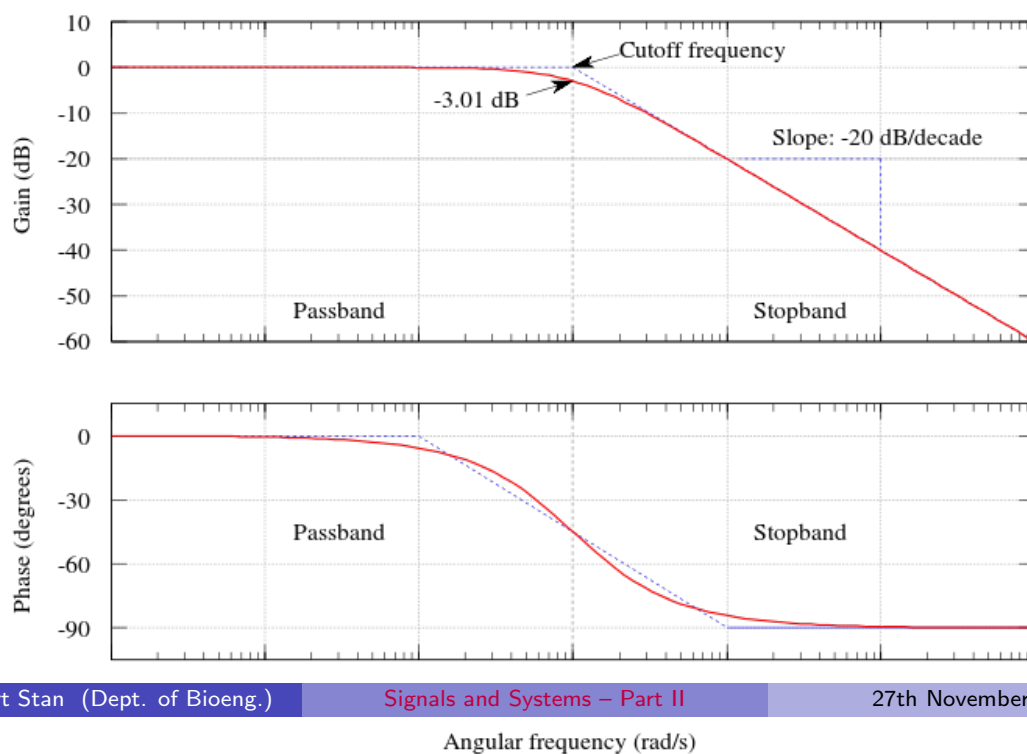
The Butterworth Filter: Passive Filter Electronic Implementation

One example of a passive circuit implementation for a third-order Butterworth filter, with $N = 3$ and $\omega_c = 1$ rad/s:

$$|H(j\omega)|^2 = \frac{1}{1 + \omega^6},$$

might look like this: