

Output synchronization in networks of cyclic biochemical oscillators

Guy-Bart Stan^{*†}, Abdullah Hamadeh^{*†}, Rodolphe Sepulchre[‡], and Jorge Gonçalves^{*}

Abstract—This paper is concerned with the global analysis of asymptotic synchronization of outputs in networks of identical oscillators. The oscillator models are assumed to possess a cyclic feedback structure. Such networks of oscillators abound in biochemistry, and are exemplified by circadian rhythm and cardiac cell networks. The main result exploits an incremental output feedback passivity property of cyclic feedback systems to prove global asymptotic output synchronization in a network composed of identical cyclic feedback systems. This result is illustrated on a network of Goodwin oscillators.

Index Terms—Global synchrony analysis, incremental dissipativity, networks of cyclic biochemical oscillators, Goodwin oscillators.

I. INTRODUCTION

Synchronization of oscillating dynamical systems is a commonly occurring phenomenon. It features in many biological networks comprised of oscillating nodes, such as those of the cardiac cells of the mammalian heart, responsible for regular heart beats or those of the neurons in the suprachiasmatic nucleus (SCN) of the hypothalamus, responsible for the generation of circadian rhythms in mammals.

Mathematically, synchronization is a convergence property for the *difference* between the solutions of interconnected systems. Convergence properties for the difference between solutions of a closed system (a system that does not interact with its environment through its inputs and outputs) are characterized by notions of *incremental stability* (see [1], [2], [3]). For open systems, the corresponding notion is *incremental dissipativity* (see [4], [5], [6], [7]).

Specifically, this paper focuses on output synchronization in networks of nodes characterized by a cyclic feedback structure. Cyclic feedback structures are typically used to model the dynamics of a chain of biochemical reactions where the final product inhibits the production of the first product in the chain whilst each intermediate product stimulates the production in the next reaction (see Figure 1). The importance of cyclic feedback structures in biology and their local stability analysis is described in [8] and [9]. In these studies, it is shown that the *secant gain condition* provides a far less conservative local stability certificate than the small gain theorem. In the more recent papers [10] and [11], this stability analysis is generalized to cyclic feedback structures

composed of output strictly passive (OSP) systems. In particular, [11] shows that the secant gain condition is a necessary as well as sufficient condition for the (global) diagonal stability of the cyclic feedback structure. Furthermore, [11] provides a constructive way to obtain a Lyapunov function for the autonomous cyclic feedback structure by considering the storage functions of the individual OSP systems.

In this paper, we analyze output synchronization in networks where each node has the cyclic feedback structure described above. To achieve this goal, we use the results in [11] and [5] to establish *incremental dissipativity* of cyclic feedback systems and to derive conditions that ensure output synchronization. Our approach aims at the characterization of the fundamental input-output properties required by cyclic feedback systems to ensure synchronization when interconnected to form a network.

The structure of this paper is as follows. In Section II, we introduce some preliminaries concerning synchronization and incremental dissipativity. Section III characterizes cyclic feedback systems and gives sufficient conditions for their incremental dissipativity. In Section IV, conditions for synchronization in networks of cyclic feedback systems are derived and discussed based on their incremental dissipativity properties. These results are illustrated on a network of Goodwin oscillators in Section V. We conclude in Section VI with a discussion of the results and of future research directions.

II. SYNCHRONIZATION AND INCREMENTAL DISSIPATIVITY

In this section, we introduce the concepts that will be used for proving (output) synchronization in networks of interconnected oscillators. Since oscillators in a network are generally connected through their inputs and outputs, it is natural to characterize them through their input-output properties in order to identify some sufficient synchronization conditions. In our approach, we consider an incremental dissipativity characterization of these oscillators that we call incremental output-feedback passivity (inc. OFP).

A. Incremental dissipativity

In this section we give a definition for *incremental dissipativity*. For a mathematical definition of dissipativity, the reader is referred to [12].

Consider an input-affine SISO system Υ represented by a state-space model of the form

$$\Upsilon \begin{cases} \dot{x} = f(x) + g(x)u, & x \in \mathbb{R}^n, u \in \mathbb{R} \\ y = h(x), & y \in \mathbb{R} \end{cases} \quad (1)$$

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where $u(t)$, $y(t)$, and $x(t)$ denote its input, output and state respectively. Let $x_a(t)$ and $x_b(t)$ be two solutions of Υ , with the corresponding input-output pairs $(u_a(t), y_a(t))$, and $(u_b(t), y_b(t))$. We denote by $\Delta x = x_a - x_b$, $\Delta u = u_a - u_b$, and $\Delta y = y_a - y_b$ the corresponding incremental variables. The system (1) is said to be incrementally dissipative if there exists a positive semi-definite incremental storage function $S_\Delta(\Delta x)$, with $S_\Delta(0) = 0$ and an incremental supply rate $w(\Delta u, \Delta y)$ such that

$$\begin{aligned} & S_\Delta(\Delta x(T^*)) - S_\Delta(\Delta x(0)) \\ & \leq \int_0^{T^*} w(\Delta u(t), \Delta y(t)) dt \end{aligned} \quad (2)$$

is satisfied for all $T^* > 0$ and along any pair of trajectories $(x_a(t), x_b(t))$. Incremental dissipativity (2) with the incremental supply rate $w(\Delta u, \Delta y) = \Delta y \Delta u$ is called *incremental passivity*. Incremental dissipativity (2) with the incremental supply rate $w(\Delta u, \Delta y) = -(\Delta y)^2 + \gamma \Delta y \Delta u$ with $\gamma \in \mathbb{R}$ is called *incremental output feedback passivity* (inc. OFP($\frac{1}{\gamma}$)). When γ is positive the system possesses an excess of incremental passivity and is said to be *incrementally output strictly passive* (inc. OSP). Following [11], $\gamma > 0$ is called the incremental secant gain of the system. On the other hand, when γ is negative the system possesses a shortage of incremental passivity and $-\frac{1}{\gamma}$ quantifies the minimum amount of proportional negative incremental output feedback required to make the system incrementally passive.

If the storage function S_Δ is additionally assumed to be differentiable, i.e. $S_\Delta \in \mathcal{C}^1$, an infinitesimal version of the dissipation inequalities associated to these concepts may be considered. For example, under differentiability assumption, the dissipation inequality (2) can be equivalently written $\dot{S}_\Delta(\Delta x) \leq w(\Delta u(t), \Delta y(t))$. For the sake of simplicity, we will assume in the rest of the paper that the storage function is differentiable and use the infinitesimal version of the dissipation inequality.

Remark 1: Passivity implies incremental passivity for linear systems, that is, if the quadratic storage $S(x) = x^T P x \geq 0$ satisfies the dissipation inequality $\dot{S} \leq yu$ then the incremental storage $S_\Delta(\Delta x) = (\Delta x)^T P \Delta x \geq 0$ satisfies the incremental dissipation inequality $\dot{S}_\Delta \leq \Delta y \Delta u$. Passivity also implies incremental passivity for a monotone increasing, static nonlinearity: if $\phi(\cdot)$ is monotone increasing, then $(s_1 - s_2)(\phi(s_1) - \phi(s_2)) = \Delta s \Delta \phi(s) \geq 0, \forall \Delta s = s_1 - s_2$. Similarly, it is easy to show that for linear systems, output strict passivity implies incremental output strict passivity with the incremental secant gain being equal to the secant gain.

B. Incremental output-feedback passivity and synchronization

The main result that links incremental output feedback passivity of nodes of a network to (output) synchronization states that if each node is inc. OFP, i.e. can be rendered incrementally passive by output feedback, and the synchronization coupling¹ between the oscillators is “strong enough” then all

¹Input-output coupling structures that allow synchronized solutions for the network are called synchronization couplings. In the linear case, synchronization coupling conditions are defined by the properties (A1), (A2) and (A3) given in Section IV. As we will see these properties allow the inc. OFP of each node to be conserved by the network.

the nodes will asymptotically output synchronize (see [13], [14], [5]).

In Section III we show that a large class of biochemical systems (called cyclic feedback systems) can be proven to be inc. OFP under some mild assumptions (see Theorem 1). We then use the above mentioned result to prove asymptotic output synchronization in networks of identical cyclic feedback systems (see Theorem 2).

Remark 2: Incremental dissipativity is an external (input-output space) property analogous to the internal (state space) property called “contraction” or “convergence” in the literature (see [14] and [3]). As originally recognized in the papers [13], [15] (with the convergence concept) and [2], [14] (with the contraction concept), output feedback convergence (i.e. convergence obtained through output feedback) (resp. output feedback contraction) is a sufficient condition to prove synchronization of coupled systems. In the input-output context, the sufficient conditions given in [13], [15] amount to assuming that the system is relative degree one and (locally) exponentially minimum phase. These conditions are strongly related to incremental output feedback passivity and are sufficient to prove local inc. OFP (see [16, Proposition 2.51]).

Remark 3: Incremental dissipativity is also strongly linked to the concept of incremental input-to-state stability (δ -ISS) since S_Δ can be seen as a δ -ISS Lyapunov function [1].

III. INCREMENTAL OUTPUT FEEDBACK PASSIVITY OF CYCLIC FEEDBACK MODELS

As in [11], we consider systems which have the cyclic feedback interconnection structure depicted in Figure 1. This class of systems arises typically when considering a sequence of biochemical reactions where the end product inhibits the rate of the first reaction while the intermediate products activate the rate of the next reaction (see [17], [8], [18] and [9]), as described by the model

$$\begin{aligned} \dot{x}_1 &= -f_1(x_1) - g_n(x_n) \\ \dot{x}_2 &= -f_2(x_2) + g_1(x_1) \\ &\vdots \\ \dot{x}_n &= -f_n(x_n) + g_{n-1}(x_{n-1}) \end{aligned}$$

where $f_i(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ and $g_i(\cdot) : \mathbb{R} \rightarrow \mathbb{R}, i = 1, \dots, n$ are increasing functions. Such a model can be seen as a cyclic feedback interconnection structure (as represented on Figure 1) with the dynamics of $H_i, i = 1, \dots, n$ defined by

$$H_i = \begin{cases} \dot{x}_i = -f_i(x_i) + u_i, & u_i \in \mathbb{R} \text{ (input)} \\ y_i = g_i(x_i), & y_i \in \mathbb{R} \text{ (output)} \end{cases} \quad (3)$$

and the following interconnection rules $u_1 = u_{ext} - y_n$ and $u_i = y_{i-1}, i = 2, \dots, n$. The corresponding input and output of this cyclic feedback system are u_{ext} and y_1 respectively. We will see that this particular choice of input-output pair plays a fundamental role in the synchronization process as it allows to establish inc. OFP of the cyclic feedback system (see Theorem 1).

A. Overview of the results

Based on the results of [11], we observe that if all the blocks H_i are output strictly passive (OSP) with a secant

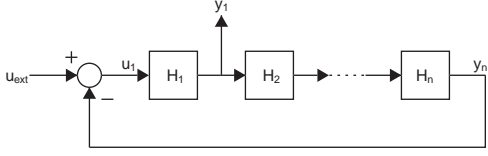


Figure 1. Cyclic feedback system.

gain $\tilde{\gamma}_i$ and a particular secant gain condition ($\tilde{\gamma}_1 \cdots \tilde{\gamma}_n < (\sec(\frac{\pi}{n}))^n$) is satisfied then the cyclic feedback system is OSP with respect to input u_{ext} and to output y_1 (see [11, Corollary 1]). As a direct corollary of this result, we can similarly prove that if all the blocks H_i are inc. OSP with an incremental secant gain γ_i and a similar incremental secant gain condition ($\gamma_1 \cdots \gamma_n < (\sec(\frac{\pi}{n}))^n$) is satisfied then the cyclic feedback system is inc. OSP with respect to input u_{ext} and to output y_1 .

Furthermore, even if the incremental secant gain condition is not satisfied, the cyclic feedback system (CFS) can be shown to be *incrementally output feedback passive*. This is intuitively clear by adding a proportional feedback gain k around H_1 and increasing the value of k (see Figure 2). The effect of the proportional feedback loop is to change the incremental secant gain of the first block. The incremental secant gain of this new first block (including the proportional feedback loop) is $\tilde{\gamma}_1 = \frac{\gamma_1}{1+k\gamma_1}$. It is clear that by increasing the value of k , the incremental secant gain condition $\tilde{\gamma}_1 \cdots \gamma_n < (\sec(\frac{\pi}{n}))^n$ will always be satisfied for large enough k (the details of this statement are included in Theorem 1).

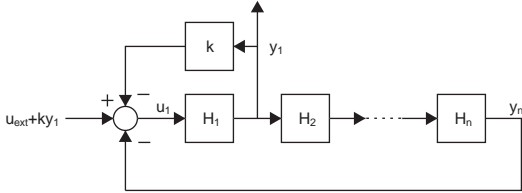


Figure 2. Equivalent system with negative feedback around H_1 .

This observation is useful for proving output synchronization in CFS networks. Indeed, coupling several CFS to form a network can always be seen as introducing a particular feedback between the inputs and the outputs of the network components. We will use this observation in Section IV to find conditions ensuring global asymptotic output synchronization in CFS networks.

B. Notation

In the next sections, we consider networks composed of N CFS. As a general convention, $i = 1, \dots, n$ will denote the index associated to a particular block H_i while $j = 1, \dots, N$ will denote the index associated to a particular CFS in the network.

The stacked vector of blocks' outputs for CFS $j = 1, \dots, N$ is denoted by $\mathbf{y}_j = (y_{1,j}, \dots, y_{n,j})^T$ ($\mathbf{y}_j \in \mathbb{R}^n$ since there are n blocks in each CFS).

The stacked vector of the first output from each CFS is denoted by Y_1 , i.e. $Y_1 = (y_{1,1}, \dots, y_{1,N})^T$ ($Y_1 \in \mathbb{R}^N$ since there are N CFS).

The stacked vector of external inputs to each CFS is denoted by U_{ext} , i.e. $U_{ext} = (u_{ext,1}, \dots, u_{ext,N})^T$ ($U_{ext} \in \mathbb{R}^N$).

C. Incremental output feedback passivity of cyclic feedback systems

In this section, based on the results presented in [11], we establish sufficient conditions under which cyclic feedback systems are incrementally output feedback passive. The main result of this section is summarized in Theorem 1.

Theorem 1: Consider the cyclic feedback system depicted in Figure 1. If each block H_i , $i = 1, \dots, n$ is inc. OSP with an incremental secant gain γ_i then the cyclic feedback system is inc. OFP($-k$) with $k \geq \frac{-1 + \gamma_1 \cdots \gamma_n (\cos(\frac{\pi}{n}))^n}{\gamma_1}$.

Proof: Assuming that all the blocks H_i are inc. OSP with an incremental secant gain γ_i , there exist incremental storage functions $V_i(\Delta x_i) \geq 0$, $V_i(0) = 0$, $i = 1, \dots, n$ such that

$$\dot{V}_1 \leq -(\Delta y_1)^2 + \gamma_1 \Delta y_1 (\Delta u_{ext} - \Delta y_n)$$

and

$$\dot{V}_i \leq -(\Delta y_i)^2 + \gamma_i \Delta y_i \Delta y_{(i-1)} \quad (4)$$

for all $i = 2, \dots, n$.

Scaling V_1 by γ_1 we obtain

$$\dot{V}_1 \leq -\frac{1}{\gamma_1} (\Delta y_1)^2 + \Delta y_1 (\Delta u_{ext} - \Delta y_n)$$

Adding and subtracting $k(\Delta y_1)^2$, $k \geq 0$, and defining $\tilde{\gamma}_1 = \frac{\gamma_1}{1+k\gamma_1}$, this can be equivalently written

$$\dot{V}_1 \leq -\frac{1}{\tilde{\gamma}_1} (\Delta y_1)^2 - \Delta y_1 \Delta y_n + k(\Delta y_1)^2 + \Delta y_1 \Delta u_{ext}$$

By rescaling V_1 by $\tilde{\gamma}_1$ we obtain

$$\dot{V}_1 \leq -(\Delta y_1)^2 - \tilde{\gamma}_1 \Delta y_1 \Delta y_n + k\tilde{\gamma}_1 (\Delta y_1)^2 + \tilde{\gamma}_1 \Delta y_1 \Delta u_{ext} \quad (5)$$

Combining (5) with (4) and using the argument of [11], the CFS storage function $V = \sum_{i=1}^n d_i V_i$ with $d_i > 0$, $i = 1, \dots, n$, can be shown to satisfy the dissipation inequality

$$\dot{V} \leq \Delta \mathbf{y}^T \left(\tilde{A}_k^T D + D \tilde{A}_k \right) \Delta \mathbf{y} + k\tilde{\gamma}_1 (\Delta y_1)^2 + \tilde{\gamma}_1 \Delta y_1 \Delta u_{ext}$$

where

$$\tilde{A}_k = \begin{pmatrix} -1 & 0 & \dots & 0 & -\tilde{\gamma}_1 \\ \gamma_2 & -1 & 0 & \ddots & 0 \\ 0 & \gamma_3 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \gamma_n & -1 \end{pmatrix}$$

and $D = \text{diag}\{d_1, \dots, d_n\}$ (see [11]).

From [11], we know that if the incremental secant gain condition

$$\tilde{\gamma}_1 \gamma_2 \cdots \gamma_n < \sec^n\left(\frac{\pi}{n}\right) \quad (6)$$

is satisfied, then the positive scalars $d_i, i = 1, \dots, n$ can be chosen such that $\tilde{A}_k^T D + D \tilde{A}_k$ is negative definite, i.e. such that $\tilde{A}_k^T D + D \tilde{A}_k \leq -\epsilon_k I_n$ with $\epsilon_k = \epsilon(k) > 0$. Using the definition of $\tilde{\gamma}_1 = \frac{\gamma_1}{1+k\gamma_1}$ we clearly see that the incremental secant gain condition (6) will always be satisfied by choosing k sufficiently large. In particular, for $k \geq \frac{-1+\gamma_1 \cdots \gamma_n (\cos(\frac{\pi}{n}))^n}{\gamma_1}$ we obtain

$$\dot{V} \leq -\epsilon_k (\Delta \mathbf{y})^T (\Delta \mathbf{y}) + \tilde{\gamma}_1 \left(k (\Delta y_1)^2 + \Delta y_1 \Delta u_{ext} \right) \quad (7)$$

with $\epsilon_k > 0$ which clearly shows that the cyclic feedback system with input u_{ext} and output y_1 is inc. OFP($-k$). ■

IV. OUTPUT SYNCHRONIZATION IN NETWORKS OF IDENTICAL CYCLIC FEEDBACK SYSTEMS

We now consider a network composed of N identical cyclic feedback systems. We assume that each CFS is inc. OFP($-k$), i.e. that each CFS is characterized by an incremental storage function which satisfies a dissipation inequality of the form (7). In particular, the incremental dissipation inequality associated with CFS $j, j = 1, \dots, N$ is

$$\dot{V}_j \leq -\epsilon_k (\Delta \mathbf{y}_j)^T (\Delta \mathbf{y}_j) + \tilde{\gamma}_1 \left(k (\Delta y_{1j})^2 + \Delta y_{1j} \Delta u_{extj} \right) \quad (8)$$

where $\epsilon_k > 0, V_j(\Delta X_j) \geq 0, V_j(0) = 0$ and $\Delta X_j = (\Delta x_{1j}^T, \dots, \Delta x_{nj}^T)^T$. As we have seen in Theorem 1, a sufficient condition that leads to (8) for CFS $j, j = 1, \dots, N$, is given by $k \geq \frac{-1+\gamma_1 \cdots \gamma_n (\cos(\frac{\pi}{n}))^n}{\gamma_1}$.

Concerning the network interconnections, we restrict ourselves to the case where the input-output coupling between the identical CFS is linear and static. The associated coupling matrix is denoted by $\Gamma \in \mathbb{R}^{N \times N}$. To ensure synchronization, we make the following assumptions on Γ :

- (A1) $\text{rank}(\Gamma) = N - 1,$
- (A2) $\Gamma + \Gamma^T \geq 0,$
- (A3) $\Gamma \mathbf{1} = \Gamma^T \mathbf{1} = \mathbf{0}.$

A. Graph interpretation of the input-output coupling assumptions

In this section, we give an interpretation of the coupling assumptions (A1), (A2), and (A3) in terms of directed graphs.

Consider a directed graph \mathcal{G} with associated weighted adjacency matrix $A = (w_{j,l}), j, l = 1, \dots, N$. Assume that the graph is simple, i.e. $w_{j,l} \geq 0$ and $w_{j,j} = 0, \forall j, l$. The corresponding weighted Laplacian matrix Γ writes $\Gamma = (\Gamma_{j,l}), j, l = 1, \dots, N$ with $\Gamma_{j,j} = \sum_{l \neq j} w_{j,l}, \forall j = 1, \dots, N$ and $\Gamma_{j,l} = -w_{j,l}, \forall j \neq l$.

The interconnection rule $U_{ext} = -\Gamma Y_1$ then corresponds to the linear consensus protocol $u_{extj} = -\sum_{l=1}^N w_{j,l} (y_{1j} - y_{1l})$ (see [19]).

Assumption (A1) holds provided that the graph is strongly connected (see [19]).

Assumption (A3) holds if the graph is balanced, i.e. if $A \mathbf{1} = A^T \mathbf{1}$ (see [20]). Furthermore, this latter property implies (A2) (see [20]).

These assumptions do not imply that Γ is symmetric which would be equivalent to assuming that the graph is undirected.

B. Output synchronization in networks of identical cyclic feedback systems

In this section, we state and prove the main result concerning asymptotic output synchronization of identical cyclic feedback systems coupled through an interconnection matrix Γ that satisfies assumptions (A1), (A2), and (A3). In Theorem 2, $\lambda_2(\Gamma_s)$ denotes the second smallest eigenvalue of the symmetric part of the interconnection matrix Γ .

Theorem 2: (Asymptotic Output Synchronization) Consider a network of N identical cyclic feedback systems linearly coupled through the interconnection matrix Γ , i.e. $U_{ext} = -\Gamma Y_1$ where Γ satisfies the assumptions (A1), (A2), and (A3). Assume that CFS $j = 1, \dots, N$ is inc. OFP($-k$), with a radially unbounded incremental storage function V_j satisfying (8), and that the network satisfies the strong coupling assumption $\lambda_2(\Gamma_s) > k$, then each network solution that exists for all $t \geq 0$ is such that $\forall i = 1, \dots, n: \forall j, l = 1, \dots, N, y_{ij}(t) = y_{il}(t)$, for $t \rightarrow +\infty$. In addition, for all bounded network solutions, the synchronized solution of a particular CFS converges to the limit set of this CFS, isolated.

Proof: To compare each CFS output with its average over all the N CFS outputs we consider the projector $\Pi = I_N - \frac{1}{N} \mathbf{1} \mathbf{1}^T$. As a consequence, $\Pi Y_1 = \Delta Y_1$ measures the difference between each output $y_{1j}, j = 1, \dots, N$ and the average output $\frac{1}{N} \sum_{j=1}^N y_{1j}$. Summing the incremental storage functions V_j given in (8) for all the CFS gives the incremental storage function $S = \sum_{j=1}^N V_j$ for the network (see [5]). Using the dissipation inequality (8), the storage function S obeys the dissipation inequality

$$\dot{S} \leq -\epsilon_k ((\Pi \otimes I_n) Y)^T ((\Pi \otimes I_n) Y) + \tilde{\gamma}_1 \left(k (\Pi Y_1)^T \Pi Y_1 + (\Pi Y_1)^T \Pi U_{ext} \right) \quad (9)$$

Since $U_{ext} = -\Gamma Y_1$ and (A3), we have $\Pi U_{ext} = -\Pi \Gamma Y_1 = -\Pi \Gamma \Pi Y_1$ so that (9) rewrites as

$$\dot{S} \leq -\epsilon_k ((\Pi \otimes I_n) Y)^T ((\Pi \otimes I_n) Y) + \tilde{\gamma}_1 \left(k (\Pi Y_1)^T \Pi Y_1 - (\Pi Y_1)^T \Gamma \Pi Y_1 \right) \quad (10)$$

Using (A1)-(A3), we have $\Pi Y_1 = Y_1 - (\frac{1}{N} \mathbf{1}^T Y_1) \mathbf{1} = \mathbf{0}$ iff $Y_1 \in \ker(\Gamma)$ and

$$(\Pi Y_1)^T \Gamma \Pi Y_1 \geq \lambda_2(\Gamma_s) (\Pi Y_1)^T \Pi Y_1 \quad (11)$$

Using (11) in (10), we obtain

$$\dot{S} \leq -\epsilon_k ((\Pi \otimes I_n) Y)^T ((\Pi \otimes I_n) Y) + \tilde{\gamma}_1 (k - \lambda_2(\Gamma_s)) (\Pi Y_1)^T \Pi Y_1$$

which yields

$$\dot{S} \leq -\epsilon_k ((\Pi \otimes I_n) Y)^T ((\Pi \otimes I_n) Y) \quad (12)$$

if $\lambda_2(\Gamma_s) > k$ (strong coupling). Application of the LaSalle invariance principle to inequality (12) proves that each network solution that exists for all $t \geq 0$ is such that $\forall i = 1, \dots, n: \forall j, l = 1, \dots, N, y_{ij}(t) = y_{il}(t)$, for $t \rightarrow +\infty$. Furthermore, since $\Gamma \mathbf{1} = \mathbf{0}$, the effect of the coupling disappears when synchrony is reached. This yields in addition to the global asymptotic output synchronization result, that for any bounded network solution the synchronized solution

of a particular CFS converges to the limit set of this CFS, isolated. ■

Combining the results of Theorem 1 and Theorem 2, we see that a sufficient condition for the asymptotic output synchronization of linearly interconnected, identical CFS is $\lambda_2(\Gamma_s) > k \geq \frac{-1 + \gamma_1 \cdots \gamma_n (\cos(\frac{\pi}{n}))^n}{\gamma_1}$.

Remark 4: If each isolated CFS is a global oscillator, the limit set of each isolated CFS is a globally attractive limit cycle.

Remark 5: Boundedness must generally be proven independently. This can be done by requiring additional assumptions such as *semipassivity*, as proposed in [13].

Remark 6: Even if all the network solutions are proven to exist for all time and to be bounded, output synchronization does not directly imply state synchronization. This will be the case under an additional incremental zero-state detectability assumption.

Remark 7: The recent paper [21] analyzes synchronization in networks of (non-identical) oscillators using an internal approach based on LMIs conditions derived from a Lyapunov function similar to our incremental storage function S . Nevertheless, this LMI formulation of the problem does not explicitly characterize the fundamental input-output properties needed to ensure synchronization.

V. ILLUSTRATION: NETWORK OF GOODWIN OSCILLATORS

As an illustration of the above theorems, we consider a network composed of N identical oscillators linearly coupled through an interconnection matrix Γ that satisfies (A1), (A2), and (A3). Each oscillator of the network is a third order, dimensionless Goodwin oscillator model [17]. The state-space realization of the dimensionless Goodwin oscillator model given in (13) is adapted from [22, Section 9.4.3] (to which the reader is referred for details of the model) and recast in the form of the cyclic feedback system studied in the previous sections. The corresponding cyclic feedback model is composed of four blocks H_i , $i = 1, \dots, 4$. In the j^{th} oscillator, the block H_{i_j} has input u_{i_j} and output y_{i_j} . The dynamical model of the j^{th} oscillator in the network is

$$\begin{aligned} H_{1_j} : \quad \dot{X}_j &= -b_1 X_j + u_{1_j}, & y_{1_j} &= X_j \\ H_{2_j} : \quad \dot{Y}_j &= -b_2 Y_j + b_2 u_{2_j}, & y_{2_j} &= Y_j \\ H_{3_j} : \quad \dot{Z}_j &= -b_3 Z_j + b_3 u_{3_j}, & y_{3_j} &= Z_j \\ H_{4_j} : & & y_{4_j} &= -\frac{1}{1 + y_{3_j}^p}, \end{aligned} \quad (13)$$

with $b_i > 0$ for $i = 1, \dots, 3$, $u_{1_j} = u_{ext_j} - y_{4_j}$, $u_{i_j} = y_{(i-1)_j}$ for $i = 2, \dots, 4$, and $j = 1, \dots, N$. The oscillators are linearly interconnected through the matrix Γ using their external inputs u_{ext_j} and outputs y_{1_j} .

All the dynamic blocks H_{i_j} , $i = 1, \dots, 3$ are linear, OSP² and inc. OSP (see Remark 1) with (incremental) secant gains given by $\frac{1}{b_1}$, 1 and 1 respectively. Furthermore, from [11, Section 6], the positive orthant of oscillator j , i.e. $\{(X_j, Y_j, Z_j) : X_j \geq 0, Y_j \geq 0, Z_j \geq 0\}$ can be shown to be invariant. For $Z_j \geq 0$, the static nonlinearity H_{4_j} is monotonically increasing with an upper bound on its

²This is easily seen using a storage function of the type $V_i = \frac{1}{2} y_{i_j}^2$, $i = 1, \dots, 3$.

derivative. It is easy to show that these latter conditions are sufficient for proving that H_{4_j} is inc. OSP in the invariant set $Z_j \geq 0$. We can also see that CFS j in (13) with input u_{ext_j} and output y_{1_j} is semipassive. To see this, first consider the uncoupled case where $u_{ext_j} = 0$. Since the output of the nonlinear element is lower bounded by -1 , $\dot{X}_j < 0$, $\forall X_j > \frac{1}{b_1}$, and thus X_j will enter the invariant set $0 \leq X_j \leq \frac{1}{b_1}$ after finite time. Consequently, it is easy to show that Y_j enters in finite time the invariant set $0 \leq Y_j \leq \frac{1}{b_1}$ and therefore that Z_j enters in finite time the invariant set $0 \leq Z_j \leq \frac{1}{b_1}$. Therefore, for $u_{ext_j} \neq 0$, the system is passive outside the state-space cube defined by the hyperplanes $X_j = Y_j = Z_j = \frac{1}{b_1}$ and the positive orthant (as can be seen by considering the storage function $V_j = \frac{1}{2} (X_j^2 + Y_j^2 + Z_j^2)$). Using [13, Lemma 1] we may thus conclude that all solutions exist for all $t \geq 0$ and are bounded.

To make the example easier, we consider $b_1 = b_2 = b_3 = b > 0$. Under this assumption, it is shown in [22] that the isolated system j possesses an equilibrium point given by $X_j = Y_j = Z_j = \xi$, where ξ is the positive solution of $\frac{1}{1 + \xi^p} = b\xi$. It is also shown that this equilibrium is locally asymptotically stable iff $p(1 - b\xi) < 8$. The variable p is termed the cooperativity coefficient and governs the rate at which the final product (in this case Z_j) inhibits the formation of X_j .

Taking $\xi = 1$ gives $b = 0.5$ and the local asymptotic stability condition $p < 16$. We can thus expect stable limit cycle oscillations to occur for $p = 17$.

The incremental secant gain of the dynamic blocks are $\gamma_1 = 2$ and $\gamma_2 = \gamma_3 = 1$ for $b = 0.5$. The incremental secant gain γ_4 of the static block H_{4_j} is equal to the maximum slope of the static nonlinearity in the invariant set $Z_j \geq 0$. This gives $\gamma_4 \approx 4.26$ for $p = 17$.

Having these values in mind, we consider the all-to-all synchronization interconnection where Γ is defined by

$$\Gamma_{j,l} = \begin{cases} (N-1)\kappa & j = l \\ -\kappa & \forall j \neq l \end{cases}$$

It can easily be checked that this all-to-all interconnection matrix Γ satisfies assumptions (A1), (A2) and (A3). Furthermore, in this case, $\lambda_2(\Gamma_s) = \kappa N$. As we proved in section IV, a sufficient condition for synchronization is

$$\lambda_2(\Gamma_s) = \kappa N > k \geq \frac{-1 + \gamma_1 \gamma_2 \gamma_3 \gamma_4 (\cos(\frac{\pi}{4}))^4}{\gamma_1} = 0.5662. \quad (14)$$

Using Theorems 1 and 2 and the boundedness property, we conclude that under condition (14) all Goodwin oscillators asymptotically synchronize and that the individual synchronized solution converges to the limit set of an isolated Goodwin oscillator (which is a limit cycle).

As an illustration, the time evolution of the outputs y_{1_j} for a network of four oscillators ($j = 1, \dots, 4$), with $\kappa = 1$ and random initial conditions, is shown in Figure 3. The corresponding state space diagram of these four oscillators is shown in Figure 4.

Simulations show that synchronization does not occur for κ sufficiently small, $\kappa = 0$ representing the uncoupled situation.

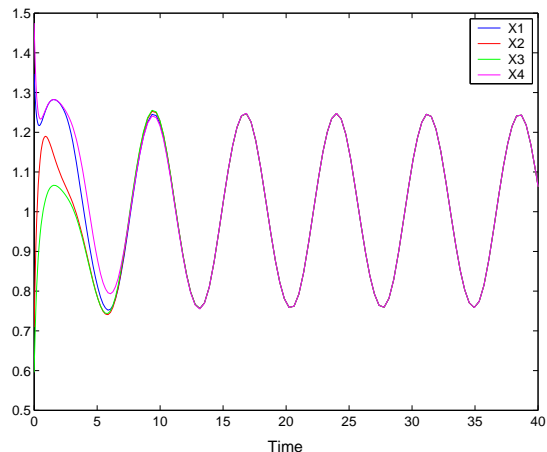


Figure 3. Time evolution of the outputs y_{1j} , $j = 1, 2, 3, 4$.

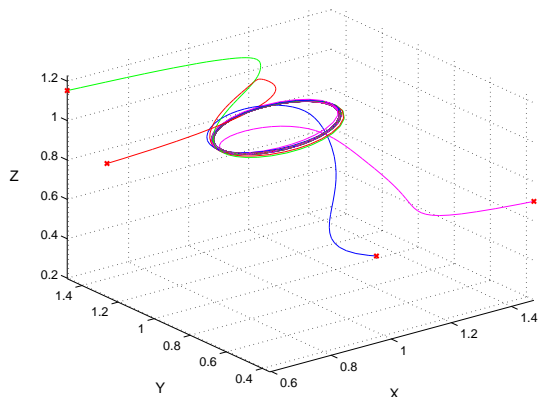


Figure 4. Limit cycle for four Goodwin oscillators.

VI. DISCUSSION

We have presented sufficient conditions for the global asymptotic output synchronization in a network of cyclic feedback systems (CFS).

The method relies on recent results in [11] which prove that a CFS is incrementally OSP (inc. OSP) if each of its constituent subsystems is inc. OSP and a particular incremental secant gain condition is satisfied. When the incremental secant gain condition is not satisfied, we have shown that CFS are nevertheless incrementally output feedback passive (inc. OFP($-k$) for k sufficiently large). Using this result, we have proven global asymptotic output synchronization in CFS networks under a strong coupling assumption.

In future work, based on extensions of dissipativity theory, we plan to investigate the conditions that lead to global boundedness of the solutions and to the existence, uniqueness and global attractivity of limit cycle oscillations for CFS. We also plan to generalize our global synchronization results to networks of nonidentical CFS.

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