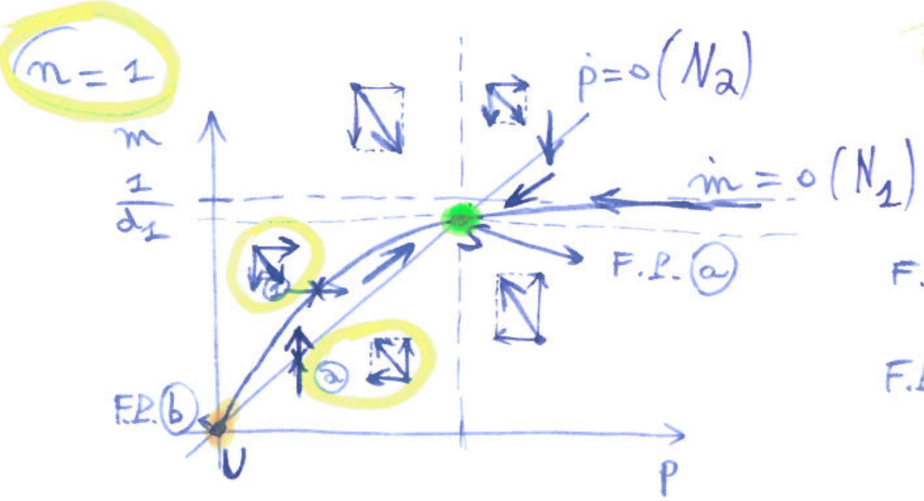
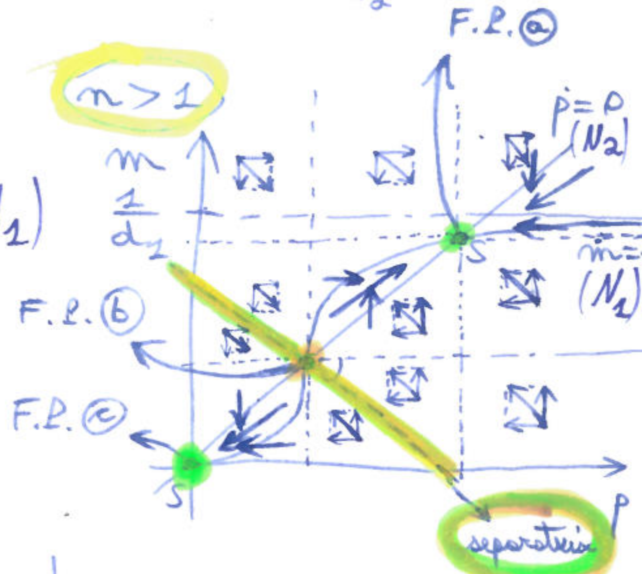
Auto-activation : Phase plane analysis

$$\begin{cases} \dot{m} = \dfrac{p^n}{K^n + p^n} - d_1 m & \rightarrow \boxed{N_1} : m = \dfrac{1}{d_1} \dfrac{p^n}{K^n + p^n} \quad (\dot{m} = 0) \\ \dot{p} = k_2 m - d_2 p & \rightarrow \boxed{N_2} : m = \dfrac{d_2}{k_2} p \quad (\dot{p} = 0) \end{cases}$$





Same reasoning here

On $\dot{m} = 0$: (i.e. on N_1)

$$\dot{p} > 0 \Leftrightarrow k_2 m - d_2 p > 0$$

$$\Leftrightarrow m > \underbrace{\frac{d_2}{k_2} p}_{N_2}$$

$\Rightarrow$ For point (1) : $\dot{p} > 0$

On $\dot{p} = 0$: (i.e. on N_2)

$$\dot{m} > 0 \Leftrightarrow \frac{p^n}{K^n + p^n} - d_1 m > 0$$

$$\Leftrightarrow m < \underbrace{\frac{1}{d_1} \frac{p^n}{K^n + p^n}}_{N_1}$$

$\Rightarrow$ For point (2) : $\dot{m} > 0$

The linearised system around the fixed point has the following form:

$$\begin{pmatrix} \dot{m} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} -d_1 & \text{linearisation of } \left(\frac{p^n}{K^n+p^n}\right)\Big|_{F.P.} \\ k_2 & -d_2 \end{pmatrix} \begin{pmatrix} m \\ p \end{pmatrix}$$

where linearisation of $\left(\frac{p^n}{K^n+p^n}\right)\Big|_{F.P.}$ is the slope of $\frac{p^n}{K^n+p^n}$ on the m vs p plot, evaluated at the considered F.P.

and $J = \begin{pmatrix} -d_1 & \text{lin}\big|_{F.P.} \\ k_2 & -d_2 \end{pmatrix}$ is the Jacobian Matrix of the linearised system

Eigenvalues of the Jacobian J

$$\det(J - \lambda I) = 0 \Leftrightarrow (\lambda + d_1)(\lambda + d_2) - k_2\, lin = 0$$

$$\Leftrightarrow \lambda^2 + (d_1 + d_2)\lambda + (d_1 d_2 - k_2\, lin) = 0$$

$$\Leftrightarrow \lambda_{\pm} = \frac{-(d_1 + d_2) \pm \sqrt{(d_1 + d_2)^2 - 4(d_1 d_2 - k_2\, lin)}}{2}$$

Let's look at the nullclines (since lin is $\approx$ the slope of one of the nullclines)

$$\alpha = \text{slope of } N_1 \text{ (i.e. } \dot{m} = 0) = \frac{1}{d_1} \cdot \underbrace{\left(\text{linearisation of } \frac{p^n}{K^n + p^n}\right)\Big|_{F.P.}}_{= lin}$$

$$\beta = \text{slope of } N_2 \text{ (i.e. } \dot{p} = 0) = \frac{d_2}{k_2}$$

- At Fixed Point (b), we see, by looking at the nullclines, that $\alpha > \beta$

$$\Leftrightarrow \frac{1}{d_1}\, lin > \frac{d_2}{k_2} > 0$$

$$\Leftrightarrow d_1 d_2 - k_2\, lin < 0$$

$$\Rightarrow \bullet\ \lambda_+ > 0 \quad \left(\text{since } (d_1 + d_2)^2 \overbrace{- 4(d_1 d_2 - k_2\, lin)}^{>0} > (d_1 + d_2)^2\right)$$

$$\bullet\ \lambda_- < 0$$

$$\Rightarrow \boxed{\text{F.P. (b) is a saddle point}}$$

. At F.P. a or at F.P. c, we see, by looking at the nullclines, that $\alpha < \beta$

$$\Leftrightarrow \quad d_1 d_2 - k_2 \, l_{in} > 0$$

$$\Rightarrow \quad (d_1 + d_2)^2 - 4 \overbrace{(d_1 d_2 - k_2 \, l_{in})}^{>0} < (d_1 + d_2)^2$$

$$\Rightarrow \quad Re(\lambda_-) < 0 \quad \text{and} \quad Re(\lambda_+) < 0$$

$\Rightarrow$ The F.P. is stable

Is it a node or a spiral?

This depends on the sign of $(d_1 + d_2)^2 - 4(d_1 d_2 - k_2 \, l_{in})$

Now: $(d_1 + d_2)^2 - 4(d_1 d_2 - k_2 \, l_{in})$

$$= (d_1^2 + 2 d_1 d_2 + d_2^2) - 4 d_1 d_2 + 4 k_2 \, l_{in}$$

$$= (d_1 - d_2)^2 + 4 k_2 \, l_{in}$$

We know l_{in} # slope of N_1 (i.e. $m = 0$)

Thus $l_{in} > 0$

$$\Rightarrow \quad \underbrace{(d_1 - d_2)^2}_{>0} + 4 \underbrace{k_2}_{>0} \underbrace{l_{in}}_{>0} > 0$$

$$\Rightarrow \quad \boxed{\text{The F.P. a and c are stable nodes}}$$