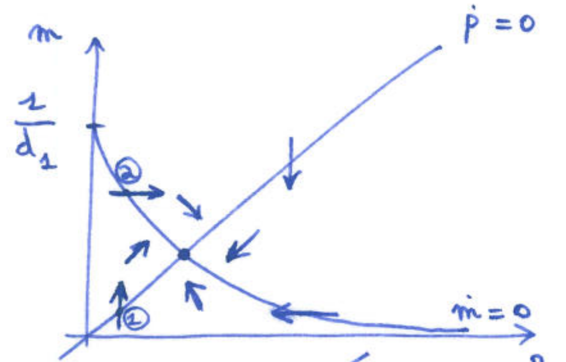
Auto-repression : Phase plane analysis

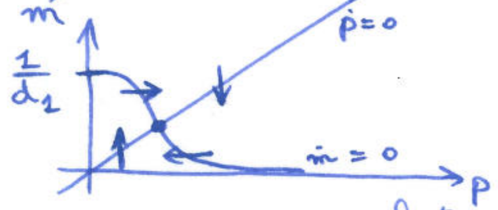
$$\begin{cases} \dot{m} = \dfrac{K^n}{K^n + p^n} - d_1 m & \rightarrow N_1 : m = \dfrac{1}{d_1} \dfrac{K^n}{K^n + p^n} \quad (\dot{m} = 0) \\ \dot{p} = k_2 m - d_2 p & \rightarrow N_2 : m = \dfrac{d_2}{k_2} p \quad (\dot{p} = 0) \end{cases}$$

$n = 1$



$$\dot{m} > 0 \Leftrightarrow \frac{K^n}{K^n + p^n} - d_1 m > 0$$
$$\Leftrightarrow m < \frac{1}{d_1} \frac{K^n}{K^n + p^n} \quad ①$$

$n > 1$



$$\dot{p} > 0 \Leftrightarrow k_2 m - d_2 p > 0$$
$$\Leftrightarrow m > \frac{d_2}{k_2} p \quad ②$$

- What is the stability of the fixed point found at $N_1 \cap N_2$? Difficult to say from the phase plane picture (even with the flow represented on the nullclines)

- Local stability analysis

not easy!

① fixed points : $N_1 \cap N_2$

$$\frac{1}{d_1} \frac{K^n}{K^n + p_{eq}^n} = \frac{d_2}{k_2} p_{eq} \Leftrightarrow \boxed{d_1 d_2 \, p_{eq} (K^n + p_{eq}^n) = k_2 K^n}$$

Not easy to solve $\forall n$!

For $n = 1$: $d_1 d_2 \, p_{eq} (K + p_{eq}) = k_2 K$ (quadratic eqn) $\to p_{eq_{1,2}} = \dots$

② Linearisation around each fixed point (if we know the fixed points!)

$\Rightarrow$ Need to linearise $\dfrac{K^n}{K^n + p^n} = f^-(p)$

$\Rightarrow$ (Taylor): compute $\dfrac{d f^-(p)}{dp}$ and evaluate it at $p = p_{eq}$ (if p_{eq} is known)

$$\frac{d f^-(p)}{dp} = - \frac{K^n}{(K^n + p^n)^2} n p^{n-1} = - \frac{n K^n p^{n-1}}{(K^n + p^n)^2}$$

(again, not easy!)

Is there an "easier" way? YES → see ~~[illegible]~~ next page

The linearised system around the fixed point has the following form:

$$\begin{pmatrix} \dot{m} \\ \dot{p} \end{pmatrix} = \underbrace{\begin{pmatrix} -d_1 & \text{linearisation of } \left(\frac{K^n}{K^n+p^n}\right) \text{ eval. at F.P.} \\ k_2 & -d_2 \end{pmatrix}}_{J} \begin{pmatrix} m \\ p \end{pmatrix}$$

~~$\tau - \Delta$ analysis~~ or eigenvalues analysis

Eigenvalues of the Jacobian matrix J

$$\det(J - \lambda I) = 0 \Leftrightarrow \begin{vmatrix} -d_1-\lambda & lin \\ k_2 & -d_2-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (d_1 + \lambda)(d_2 + \lambda) - k_2 \cdot lin = 0$$

$$\Leftrightarrow \boxed{\lambda^2 + (d_1 + d_2)\lambda + (d_1 d_2 - k_2\, lin) = 0}$$

$$\boxed{\lambda_\pm = \frac{-(d_1 + d_2) \pm \sqrt{(d_1 + d_2)^2 - 4(d_1 d_2 - k_2\, lin)}}{2}}$$

Let's look at the nullclines (since linearisation is related to the slopes of the nullclines at the F.P.)

$$:= \text{slope of } N_1 \ (\text{i.e. } \dot{m} = 0)\Big|_{F.P.} = \frac{1}{d_1} \cdot \left(\text{linearisation of } \frac{K^n}{K^n+p^n} \text{ at F.P.}\right)$$

$$= \text{slope of } N_2 \ (\text{i.e. } \dot{p} = 0)\Big|_{F.P.} = \frac{d_2}{k_2}$$

By looking at the phase plane, we know that $\alpha < \beta$ (actually $\alpha < 0$ and $\beta > 0$)

So we have:

$$\alpha = \frac{1}{d_1} \cdot lin < \frac{d_2}{k_2} = \beta \qquad \text{with } d_1 > 0 \ \& \ k_2 > 0$$

Thus: $\boxed{d_1 d_2 - k_2\, lin > 0} \Rightarrow \Re(\lambda_-) < 0$ and $\Re(\lambda_+) < 0$

$\Rightarrow$ The F.P. is stable

Is it a node or a spiral?

Node or spiral?

This depends on the sign of

$$\boxed{\gamma = (d_1 + d_2)^2 - 4\,(\underbrace{d_1 d_2 - k_2\, lin}_{>0})}$$

If $\gamma > 0 \Rightarrow$ node (stable)

If $\gamma < 0 \Rightarrow$ spiral (stable)

Now $\gamma = d_1^2 + 2 d_1 d_2 + d_2^2 - 4 d_1 d_2 + 4 k_2\, lin$

$\Leftrightarrow \gamma = (d_1 - d_2)^2 + 4 k_2\, lin$

with lin # slope of N_1 (i.e. $\dot{m} = 0$)

$\Rightarrow \boxed{lin < 0}$

γ can be either > 0 or < 0 depending on $|lin|$ and on k_2

For example: if $d_1 = d_2$: $\gamma = 4 k_2\, lin < 0$

$\Rightarrow$ stable spiral

if $d_1 \gg d_2$ and $|k_2\, lin| \ll 1$

$\Rightarrow \gamma > 0 \Rightarrow$ stable node