Constructive Synchronization of Networked Feedback Systems

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***Abstract*— This paper is concerned with global asymptotic output synchronization in networks of identical feedback systems. Using an operator theoretic approach based on an incremental small gain theorem, the method reformulates the synchronization problem as one of achieving incremental stability using a coupling operator that plays the role of an incrementally stabilizing feedback. In this way, conditions on static or dynamic coupling operators that achieve output synchronization of nodes of arbitrary structure are derived. These conditions lead to a methodology for the construction of coupling architectures that ensure output synchronization of a wide range of systems. The result is illustrated for a network of biochemical oscillators.**

I. Introduction

This paper presents a sufficient condition for output synchronization in networks of interconnected dynamical systems and provides a constructive means of establishing network interconnection structures that will result in output synchronization.

Output synchronization is a stability property for the difference between the outputs of interconnected systems and can be studied using concepts stemming from incremental stability [1] or contraction theory [2], [3]. Viewed in another way, we can determine whether two coupled systems synchronize by studying the asymptotic attractivity and stability of a synchronization manifold on which corresponding states of the interconnected systems have a common value. Several works have examined the *local* stability of the synchronization manifold. In these, the general approach has been to use transverse Lyapunov exponents [4] and Master Stability Functions [5], [6] to show that under certain coupling conditions the components of the trajectories transverse to the synchronization manifold are stable in a neighborhood of the manifold. The key observation of [2], [3] is that proving asymptotic state synchronization requires showing that the differences between corresponding states of the coupled systems satisfy a contraction property. This can be done by constructing a Lyapunov function that operates on these *incremental* signals. In the case of identical systems, where outputs are continuous functions of the states, asymptotic state synchronization implies asymptotic output synchronization.

Practical examples of synchronization phenomena abound in physical and biochemical systems, including cardiac pacemaking and the maintenance of circadian rhythms in many organisms. In [7], a model of the mammalian circadian pacemaker is presented, composed of a network of nodes, each consisting of three-dimensional cyclic feedback systems (CFS), the first outputs of which diffuse throughout the network to other nodes. Simulations showed that a coupling mechanism composed of a first-order dynamical system and an all-to-all network topology resulted in synchronization of the nodal states.

In our previous work [8], [9], we presented a constructive method showing that, under a strong linear static coupling condition, such cyclic systems will synchronize. The method assumed that each CFS was composed of a ring of incrementally output strictly passive subsystems, a quality analogous to the output strict passivity notion of dissipative systems [10] that operates on incremental system signals. It was shown that the linear static coupling acted as a passifying incremental feedback by increasing the degree of incremental output strict passivity of individual CFS subsystems (quantified by the so-called incremental secant gain). In this way, the CFS network was made incrementally diagonally stable by reducing the product of the incremental secant gains below a threshold, as [11] does for demonstrating CFS stability. In [12], an input/output method was used to extend [8] by determining synchronization conditions for nodes more general than CFS, using the same linear static coupling mechanism and the incremental diagonal stability idea.

In this paper, we extend the class of nodes that can be synchronized to systems composed of an $\mathcal{L}_{2e}$ operator with a unity-gain feedback. The class of coupling structures that are considered in this paper is also extended beyond those in [8], [12] to include operators on $\mathcal{L}_{2e}$. Taken together, the forward paths of the nodes and the network coupling are shown to compose a feedback system. By finding a coupling mechanism that ensures that this feedback system amplifies signals orthogonal to the synchronization manifold by no more than unity, this paper uses an incremental small gain theorem to establish incremental $\mathcal{L}_2$ output stability and hence output synchronization. Finding such a coupling mechanism is an incremental variant of the classical sub-optimal $\mathcal{H}_\infty$ control problem (see, e.g. [13], [14]). This provides a way of finding nodal coupling mechanisms that ensure synchronization and constitutes the main design contribution of this paper.

There are several contributions in this paper beyond [8] and [12]. In both these works, proving synchronization involved taking advantage of the incremental dissipativity properties of the nodal subsystems to prove incremental diagonal stability, which required the construction of a diagonal Lyapunov function. This is not straightforward, nor always possible for nodes of arbitrary structure interconnected using general coupling mechanisms. The incremental small gain theorem approach herein simplifies the derivation of sufficient synchronization conditions.

It was also previously assumed in [8], [12] that subsystems of each node were directly coupled to their corresponding subsystems in other nodes. This is a fundamental limitation imposed by the method used in [8] because the network was posed as an incrementally output feedback passive system and incremental stability was achieved by using a linear static coupling as an incremental output feedback that eliminated the network's shortage of incremental passivity. This formulation of the problem required the incremental

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system representing the network to be of relative degree one, which, in terms of incremental signals implied that coupling signals could only be exchanged between corresponding nodal subsystems. We relax this condition to allow arbitrary coupling. We also relax the restriction that the inter-nodal coupling must be linear and static and allow the coupling to take the form of any $\mathcal{L}_{2e}$ operator. This enables us to analyze models of dynamically coupled circadian oscillators such as [7].

This paper is organized as follows. After detailing notation, the general forms of the network nodes and the coupling mechanism we assume are described. The tools that will be employed in the later sections of the paper are then formally defined, followed by the main theorem. We then discuss applications of the main theorem prior to presenting an example that demonstrates the paper's contribution.

II. Notation

This paper will consider networks composed of N nodes. As a general convention, $j = 1, \cdots, N$ will denote the index associated with a particular node. With reference to Figure 1, the forward path of each node has two input sources $\mathbf{w}_j \in \mathbb{R}^m, \mathbf{u}_j \in \mathbb{R}^n$ and two output sources $\mathbf{z}_j \in \mathbb{R}^p, \mathbf{y}_j \in \mathbb{R}^n$.

- The vector of the outputs of the j^{th} node is given by $\mathbf{y}_j = \left[\begin{array}{ccc} y_{1_j} & \cdots & y_{n_j} \end{array}\right]^*$, and the vectors $\mathbf{w}_j \in \mathbb{R}^m, \mathbf{u}_j \in \mathbb{R}^n, \mathbf{z}_j \in \mathbb{R}^p$ are similarly defined.
- The vector $Y_i = \left[\begin{array}{ccc} y_{i_1} & \cdots & y_{i_N} \end{array}\right]^* \in \mathbb{R}^N$ is a vector of the i^{th} element of the vectors $\mathbf{y}_j$, $\forall j \in 1, \cdots, N$. The vectors $W_i, V_i, U_i, Z_i \in \mathbb{R}^N$ are similarly defined.
- The vector of all outputs y_{i_j} is $Y = \left[\begin{array}{ccc} Y_1^* & \cdots & Y_n^* \end{array}\right]^* \in \mathbb{R}^{Nn}$. The vectors $W \in \mathbb{R}^{Nm}, U \in \mathbb{R}^{Nn}, Z \in \mathbb{R}^{Np}$ are similarly defined.

$I_r \in \mathbb{R}^{r \times r}$ is the r-dimensional identity matrix, $\mathbf{1}_r$ ($\mathbf{0}_r$) is a column vector of ones (zeros) in $\mathbb{R}^r$, $\mathbf{1}_{r \times r}$ ($\mathbf{0}_{r \times r}$) is a matrix of ones (zeros) in $\mathbb{R}^{r \times r}$.

We define the operator Π as $\Pi = I_N - \frac{1}{N}\mathbf{1}_{N \times N}$, $N \in \mathbb{Z}_+$. As described in [15], [8], the operator Π measures the lack of consensus between the elements of a vector. For example, for the vector $Y_i \in \mathbb{R}^N$ the k^{th} element of the vector ΠY_i is the difference between the k^{th} element of Y_i and the average of all the elements of Y_i. Note that $\Pi^* \Pi = \Pi$. We define $\tilde{\Pi}_r = I_r \otimes \Pi$.

We use the notation $\| \cdot \|$ to denote the $\mathcal{L}_2-$norm of a signal. The space $\mathcal{L}_2^r$ is the space of square-integrable signals on the domain $[0, \infty)$ of dimension r. The set $\mathcal{L}_{2e}^r$ is the extended space of square-integrable r-dimensional vectors on the domain $[0, T], \forall T > 0$. The notation $\| \cdot \|_T$ denotes the $\mathcal{L}_2$ norm of a signal restricted to the domain $[0, T]$.

III. Characterization of Network Nodes and Nodal Coupling Structure

Let $\mathbf{F}_j : \mathcal{L}_{2e}^{m+n} \to \mathcal{L}_{2e}^{p+n}$ be a strongly causal, locally Lipschitz continuous map [16] from input vector $\left[\begin{array}{cc} \mathbf{w}_j^* & \mathbf{u}_j^* \end{array}\right]^* \in \mathbb{R}^{m+n}$ to output vector $\left[\begin{array}{cc} \mathbf{z}_j^* & \mathbf{y}_j^* \end{array}\right]^* \in \mathbb{R}^{p+n}$, where $\mathbf{w}_j \in \mathbb{R}^m$, $\mathbf{u}_j \in \mathbb{R}^n$, $\mathbf{z}_j \in \mathbb{R}^p$, $\mathbf{y}_j \in \mathbb{R}^n$.

A network of N identical interconnected systems will be considered, where each node, with index $j = 1, \cdots, N$, is a feedback system of the form shown in Figure 1, defined as

$$\left[\begin{array}{c} \mathbf{z}_j \\ \mathbf{y}_j \end{array}\right] = \mathbf{F}_j\left(\left[\begin{array}{c} \mathbf{w}_j \\ \mathbf{u}_j \end{array}\right]\right), \qquad \mathbf{u}_j = \mathbf{y}_j \tag{1}$$

The maps $\mathbf{F}_j$ and the input/output signals from all nodes will be combined so that the operator $\mathbf{F} : \mathcal{L}_{2e}^{N(m+n)} \to \mathcal{L}_{2e}^{N(p+n)}$ defines the map

$$\left[\begin{array}{c} Z \\ Y \end{array}\right] = \mathbf{F}\left(\left[\begin{array}{c} W \\ U \end{array}\right]\right) \tag{2}$$

and the nodes collectively satisfy the relation

$$U = Y \tag{3}$$

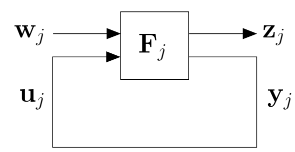


Fig. 1. The j^{th} node coupled using its input $\mathbf{w}_j$ and output $\mathbf{z}_j$.

The node j outputs signals to other nodes in the network through its output vector $\mathbf{z}_j$ and receives signals from them though its external input vector $\mathbf{w}_j$. It is assumed the nodal coupling obeys the following properties:

(P1) A causal, locally Lipschitz continuous operator $\mathbf{C} : \mathcal{L}_{2e}^{Np} \to \mathcal{L}_{2e}^{Nm}$ maps the output vector $Z \in \mathbb{R}^{Np} := [Z_1^*, \cdots, Z_p^*]^*$ to a signal $V \in \mathbb{R}^{Nm} := [V_1^*, \cdots, V_m^*]^*$.

(P2) Outputs V_i of the coupling map $\mathbf{C}$ in (P1) diffuse through the network to the external inputs W_i of the different nodes via a weighted directed graph $\mathcal{G}^i$ (defined below), specific for every $i = 1, \cdots, m$ and representing the different diffusions of the species V_i to the network nodes. Associated with each $\mathcal{G}^i$ is the Laplacian matrix $\Gamma_i \in \mathbb{R}^{N \times N}$ that defines the topology of $\mathcal{G}^i$ by the mapping $W_i = -\Gamma_i V_i$. Defining $\tilde{\Gamma} = \text{diag}\{\Gamma_1, \cdots, \Gamma_m\}$ we have

$$W = -\tilde{\Gamma} V = -\tilde{\Gamma}\mathbf{C}(Z) \tag{4}$$

Graph $\mathcal{G}^i = \{\mathcal{A}^i, \mathcal{D}^i\}$ in (P2) has the definitions

Definition 1 (Weighted Adjacency Matrix): A weighted adjacency matrix $\mathcal{A}^i = \left\{\rho_{j,l}^i\right\}$, $j, l = 1, \cdots, N$, $\mathcal{A}^i \in \mathbb{R}^{N \times N}$, is a positive matrix where $\rho_{j,l}^i$ represents the weight of the edge from node l to node j. We assume that the graph is simple, i.e. $\rho_{j,l}^i \geq 0$, $\forall j \neq l$ and $\rho_{j,j}^i = 0$, $\forall j, l.\diamondsuit$

Definition 2 (Degree Matrix): The degree matrix $\mathcal{D}^i$ associated with the adjacency matrix $\mathcal{A}^i$ is a diagonal matrix $\mathcal{D}^i = \text{diag}\{\delta_j^i\}$, $j = 1, \cdots, N$, $\mathcal{D}^i \in \mathbb{R}^{N \times N}$ with $\delta_j(i) = \sum_{\substack{l=1 \\ l \neq j}}^N \rho_{j,l}^i.\diamondsuit$

Definition 3 (Laplacian Matrix): The weighted Laplacian matrix Γ_i associated with the adjacency matrix $\mathcal{A}^i$ is defined as $\Gamma_i = \mathcal{D}^i - \mathcal{A}^i = \left\{\Gamma_{j,l}^i\right\}$, $j, l = 1, \cdots, N$ with $\Gamma_{j,j}(i,k) = \sum_{\substack{l=1 \\ l \neq j}}^N \rho_{j,l}$, $\forall j = 1, \cdots, N$ and $\Gamma_{j,l}(i,k) = -\rho_{j,l}^i$, $\forall j \neq l.\diamondsuit$

The interconnection rule $W_i = -\Gamma_i V_i$ then corresponds to the linear consensus protocol $w_{i_j} = -\sum_{l=1}^N \rho_{j,l}^i \left(v_{i_j} - v_{i_l}\right)$ (see [17]). We make the following assumptions on Γ_i:

(A1) $\text{rank}(\Gamma_i) = N - 1$
(A2) $\Gamma_i + \Gamma_i^* \geq 0$
(A3) $\Gamma_i \mathbf{1}_N = \Gamma_i^* \mathbf{1}_N = \mathbf{0}_N$

Assumption (A1) holds provided that the graph is strongly connected (see [17]). Assumption (A3) holds if the graph is balanced, i.e. if $\mathcal{A}\mathbf{1}_N = \mathcal{A}^*\mathbf{1}_N$ (see [18]). Furthermore, this latter property implies (A2) (see [18]). Note that these assumptions do not imply that Γ_i is symmetric which would be equivalent to assuming an undirected graph.

For a matrix $\Gamma_i \in \mathbb{R}^{N\times N}$, we define $\lambda_{r_i} = \lambda_{r_i}(\Gamma_i)$ (for $i = 1, \cdots, N$) as the r^{th} smallest eigenvalue of the matrix $\frac{1}{2}(\Gamma_i + \Gamma_i^*)$, the symmetric part of the matrix Γ_i.

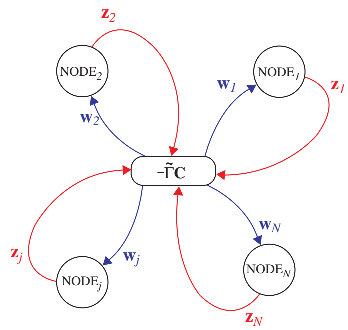


Fig. 2. The N nodes are coupled using outputs $\mathbf{z}_j \in \mathbb{R}^p$ and inputs $\mathbf{w}_j \in \mathbb{R}^m$. The network coupling is composed of a map $W = -\tilde{\Gamma}V = -\tilde{\Gamma}\mathbf{C}(Z)$, where $\mathbf{C} : \mathcal{L}_{2e}^{Np} \to \mathcal{L}_{2e}^{Nm}$, and $\tilde{\Gamma}$ is a collection of Laplacian matrices Γ_i which determine the network topology by mapping signals V_i (not shown) to the nodal external inputs W_i (not shown).

Consider a network of N nodes of the form satisfying (2), (3), coupled using an interconnection structure that satisfies (P1), (P2). The interconnection of (2) and the coupling (4) creates the closed loop system composed of the map $\tilde{\mathbf{F}}$. Therefore the interconnected network composed of (2), (3) and the coupling (4) can be regarded as the feedback system

$$Y = \tilde{\mathbf{F}}(U) \tag{5}$$
$$U = Y \tag{6}$$

which is illustrated in Figure 3.

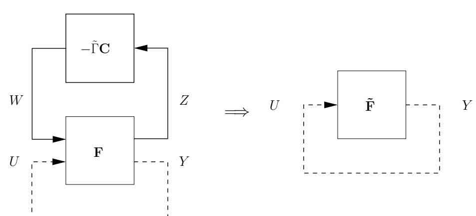


Fig. 3. The network of interconnected nodes is represented by the feedback system (5), (6) resulting from the interconnection of (2), (3) and (4).

IV. Main results

The approach that will be taken in analyzing synchronization involves placing a metric on the difference between the corresponding outputs of the network nodes y_{i_j} and then identifying the coupling conditions needed for these differences to reduce to zero. To measure consensus between network signals we shall make use of the projector matrix Π. The following notion of incremental finite-gain $\mathcal{L}_2$ stability will be used to characterize the convergence properties of the network nodes.

Definition 4 (Incremental finite-gain $\mathcal{L}_2$ stability): The map $\tilde{\mathbf{F}}$ from the input signal $U \in \mathbb{R}^{Nn}$ to the output signal $Y \in \mathbb{R}^{Nn}$ is an incrementally finite-gain $\mathcal{L}_2$-stable map if there exist $\tilde{\gamma}, \tilde{\eta} \geq 0$ such that the inequality

$$\|\tilde{\Pi}_n Y\|_T \leq \tilde{\gamma}\|\tilde{\Pi}_n U\|_T + \tilde{\eta} \tag{7}$$

is satisfied for all $T \geq 0$.

Definition 5 (Incremental $\mathcal{L}_2$ gain): Suppose the map $\tilde{\mathbf{F}}$ is incrementally finite-gain $\mathcal{L}_2$ stable in the sense of Definition 4. The incremental $\mathcal{L}_2$ gain of the map $\tilde{\mathbf{F}}$ is defined as

$$\tilde{\gamma}(\tilde{\mathbf{F}}) := \inf\{\tilde{\gamma} | \exists \tilde{\eta} \geq 0 \text{ such that (7) holds}\}$$

Characterizing the incremental finite-gain $\mathcal{L}_2$ stability of the mapping $\tilde{\mathbf{F}}$ allows us to analyze signals that are orthogonal to the *synchronization manifold* $y_{i_1} = \cdots = y_{i_N}, \forall i$. In Theorem 1 the small gain theorem will be applied to the map $\tilde{\mathbf{F}}$ under the relation (3) to provide a sufficient condition for the $\mathcal{L}_2$ stability of the signal $\tilde{\Pi}_n Y$, thus leading to output synchronization, which is defined as follows:

Definition 6 (Output synchronization): The outputs of a collection of N nodes of the form (1) are said to be output synchronized when $y_{i_j} = y_{i_k}, \forall i \in \{1, \cdots, n\}, \forall j, k \in \{1, \cdots, N\}$.

The signal $\tilde{\Pi}_n Y$ provides a measure of the difference between network outputs. This can be seen by noting that

$$Y^* \tilde{\Pi}_n^* \tilde{\Pi}_n Y = \frac{1}{2N} \sum_{i=1}^{n} \sum_{j=1}^{N} \sum_{k=1}^{N} (y_{i_j} - y_{i_k})^2$$

Therefore proving that the signal $\tilde{\Pi}_n Y \in \mathcal{L}_2^{Nn}$ and assuming that outputs Y are uniformly continuous in time would imply that Y will tend to the synchronization manifold as signals transverse to the manifold decay to zero. Formally, this means that $\lim_{t\to\infty} |y_{i_j}(t) - y_{i_k}(t)| = 0, \forall i = 1, \cdots, m$ and $\forall j, k \in 1, \cdots, N$ so that, in the limit $t \to \infty$, the network nodes' outputs synchronize in the sense of Definition 6.

Theorem 1: Consider a network in which N identical nodes of the form given by (1) are interconnected with a coupling structure that satisfies (P1), (P2), leading to the system (5), (6). If the map $\tilde{\mathbf{F}}$ is incrementally finite-gain $\mathcal{L}_2$ stable in the sense of Definition 4 with incremental $\mathcal{L}_2$ gain $\tilde{\gamma}(\tilde{\mathbf{F}}) < 1$, and if outputs $y_{i_j}(t)$ for $j = 1, \cdots, N$ and $i = 1, \cdots, n$ are uniformly continuous on $[0, \infty)$ then the network outputs are such that $\lim_{t\to\infty} |y_{i_j}(t) - y_{i_k}(t)| = 0, \forall i = 1, \cdots, n, \forall j, k = 1, \cdots, N$ thus achieving output synchronization in the sense of Definition 6 as $t \to \infty$.

Proof: From the strong causality of $\mathbf{F}$, the network interconnection is well-posed by Theorem 4.1 of [16] and existence and uniqueness of solutions is thus guaranteed.

The remainder of the proof is an application of the small gain theorem to the incremental signal $\tilde{\Pi}_n Y$. Since $\tilde{\mathbf{F}}$ is incrementally finite-gain $\mathcal{L}_2$ stable it follows that $\|\tilde{\Pi}_n Y\| \leq \tilde{\gamma}(\tilde{\mathbf{F}})\|\tilde{\Pi}_n U\| + \tilde{\eta}$. From the condition $U = Y$ in (6) we have$\|\tilde{\Pi}_n Y\| \leq \tilde{\gamma}(\tilde{\mathbf{F}})\|\tilde{\Pi}_n Y\| + \tilde{\eta}$. The condition $\tilde{\gamma}(\tilde{\mathbf{F}}) < 1$ then ensures that $\tilde{\Pi}_n Y \in \mathcal{L}_2^{Nn}$ since $\|\tilde{\Pi}_n Y\| < \frac{1}{1-\tilde{\gamma}(\tilde{\mathbf{F}})}\tilde{\eta}$. Since outputs y_{i_j} are uniformly continuous, we can invoke Barbalat's lemma (see [19]) to prove that $\lim_{t\to\infty}(\tilde{\Pi}_n Y)^*(\tilde{\Pi}_n Y) = 0$ which is true if and only if $\lim_{t\to\infty} |y_{i_j}(t) - y_{i_k}(t)| = 0, \forall i = 1, \cdots, n, \forall j, k = 1, \cdots, N$. ■

Remark 1: The condition that outputs y_{i_j} be uniformly continuous can be met if the outputs are bounded and continuous. The continuity condition can be met, as discussed in [20], for the example of operators $\mathbf{F}$ having a state-space realization wherein the time derivative of the state is a locally Lipschitz function of the state and where the output is a continuous function of the state.

Theorem 1 poses the problem of synchronizing a network of feedback systems as one of finding a coupling structure that makes the incremental $\mathcal{L}_2$ gain of $\tilde{\mathbf{F}}$ smaller than unity. In the case where the network nodes are not connected ($\tilde{\Gamma} = \mathbf{0}_{Nm\times Nm}$), the map $\tilde{\mathbf{F}}$ will generally not satisfy the conditions of Theorem 1 by having too large (or infinite) an incremental $\mathcal{L}_2$ gain. In the following, it will be demonstrated how the coupling can be used to reduce the incremental $\mathcal{L}_2$ gain to meet the synchronization conditions of Theorem 1.

V. Linear dynamic coupling

In this section we shall give sufficient conditions for synchronization in the sense of Definition 6 to take place when the network coupling is composed of an LTI dynamical system and when network nodes are composed of multiple subsystems, at least one of which is LTI. We shall show that in the case where the LTI subsystem inputs and outputs are used to couple the nodes, an LTI coupling can be chosen so as to render the nodes incrementally stable.

Consider a network of N nodes of the form (1), each composed of a cascade of two subsystems H and G with unity gain feedback, as shown in Figure 4. Let $\mathbf{w}_j \in \mathbb{R}^m$, $\mathbf{z}_j \in \mathbb{R}^p$, $\mathbf{r}_j \in \mathbb{R}^q$, $\mathbf{u}_j$ and $\mathbf{y}_j \in \mathbb{R}^n$. The maps $H : \mathcal{L}_{2e}^{m+n} \to \mathcal{L}_{2e}^{p+q}$ and $G : \mathcal{L}_{2e}^{q} \to \mathcal{L}_{2e}^{n}$ (respectively $\mathbf{H} : \mathcal{L}_{2e}^{N(m+n)} \to \mathcal{L}_{2e}^{N(p+q)}$ and $\mathbf{G} : \mathcal{L}_{2e}^{Nq} \to \mathcal{L}_{2e}^{Nn}$) are such that

$$\begin{array}{ll} \begin{bmatrix} \mathbf{z}_j \\ \mathbf{r}_j \end{bmatrix} = H\left(\begin{bmatrix} \mathbf{w}_j \\ \mathbf{u}_j \end{bmatrix}\right) & \begin{bmatrix} Z \\ R \end{bmatrix} = \mathbf{H}\left(\begin{bmatrix} W \\ U \end{bmatrix}\right) \\ \mathbf{y}_j = G(\mathbf{r}_j) & Y = \mathbf{G}(R) \\ \mathbf{u}_j = \mathbf{y}_j & U = Y \end{array} \tag{8}$$

with R being defined analogously to signals Y, U, W, Z in Section II.

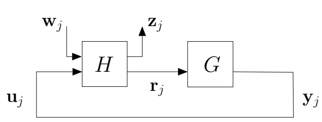


Fig. 4. A network node composed of a cascade of an LTI system H and an $\mathcal{L}_{2e}$ operator G, with unity gain feedback.

Assumption 1: The map H is strongly causal, is locally Lipschitz continuous, is linear with transfer matrix $\bar{H}(s) \in \mathbb{C}^{p+q\times m+n}$ and has the decomposition

$$\bar{H}(s) = \begin{bmatrix} \bar{H}_{z_w}(s) & \bar{H}_{z_u}(s) \\ \bar{H}_{r_w}(s) & \bar{H}_{r_u}(s) \end{bmatrix}$$

where $\bar{H}_{z_w}(s) \in \mathbb{C}^{p\times m}$, $\bar{H}_{z_u}(s) \in \mathbb{C}^{p\times n}$, $\bar{H}_{r_w}(s) \in \mathbb{C}^{q\times m}$, $\bar{H}_{r_u}(s) \in \mathbb{C}^{q\times n}$. The corresponding composite map $\bar{\mathbf{H}}(s) = \bar{H}(s) \otimes I_N$ is such that

$$\bar{\mathbf{H}}(s) = \begin{bmatrix} \bar{\mathbf{H}}_{z_w}(s) & \bar{\mathbf{H}}_{z_u}(s) \\ \bar{\mathbf{H}}_{r_w}(s) & \bar{\mathbf{H}}_{r_u}(s) \end{bmatrix} = \begin{bmatrix} \bar{H}_{z_w}(s) \otimes I_N & \bar{H}_{z_u}(s) \otimes I_N \\ \bar{H}_{r_w}(s) \otimes I_N & \bar{H}_{r_u}(s) \otimes I_N \end{bmatrix}$$

and satisfies the Laplace domain relation

$$\begin{bmatrix} \bar{Z}(s) \\ \bar{R}(s) \end{bmatrix} = \bar{\mathbf{H}}(s) \begin{bmatrix} \bar{W}(s) \\ \bar{U}(s) \end{bmatrix} \tag{9}$$

Using Assumption 1, the following proposition gives conditions for the existence of a mapping from the vector of incremental input signals to the map $\bar{\mathbf{H}}(s)$, given by $\begin{bmatrix} \tilde{\Pi}_m \bar{W}(s)^* & \tilde{\Pi}_n \bar{U}(s)^* \end{bmatrix}^*$ to the vector of incremental output vectors $\begin{bmatrix} \tilde{\Pi}_p \bar{Z}(s)^* & \tilde{\Pi}_q \bar{R}(s)^* \end{bmatrix}^*$.

Proposition 1: Map $\mathbf{H} : \mathcal{L}_{2e}^{N(m+n)} \to \mathcal{L}_{2e}^{N(p+q)}$, with transfer function $\bar{\mathbf{H}}(s)$ is, under Assumption 1, such that

$$\begin{bmatrix} \tilde{\Pi}_p \bar{Z}(s) \\ \tilde{\Pi}_q \bar{R}(s) \end{bmatrix} = \bar{\mathbf{H}}(s) \begin{bmatrix} \tilde{\Pi}_m \bar{W}(s) \\ \tilde{\Pi}_n \bar{U}(s) \end{bmatrix} \tag{10}$$

Proof: The proof follows from the fact that

$$\begin{aligned} &\begin{bmatrix} I_p \otimes \Pi & \mathbf{0}_{Np\times Nq} \\ \mathbf{0}_{Nq\times Np} & I_q \otimes \Pi \end{bmatrix} \begin{bmatrix} \bar{H}_{z_w}(s) \otimes I_N & \bar{H}_{z_u}(s) \otimes I_N \\ \bar{H}_{r_w}(s) \otimes I_N & \bar{H}_{r_u}(s) \otimes I_N \end{bmatrix} \\ &= \begin{bmatrix} \bar{H}_{z_w}(s) \otimes I_N & \bar{H}_{z_u}(s) \otimes I_N \\ \bar{H}_{r_w}(s) \otimes I_N & \bar{H}_{r_u}(s) \otimes I_N \end{bmatrix} \begin{bmatrix} I_m \otimes \Pi & \mathbf{0}_{Nm\times Nn} \\ \mathbf{0}_{Nn\times Nm} & I_n \otimes \Pi \end{bmatrix} \end{aligned}$$

which is obtained from the properties of the Kronecker product applied to the transfer function $\bar{\mathbf{H}}(s)$. ■

We assume that networks of nodes of the form (8) are interconnected using a coupling scheme that satisfies (P1), (P2), and now restrict the analysis to the case where the map $\mathbf{C}$ is an LTI system with corresponding transfer function $\bar{\mathbf{C}}(s)$. We will next examine the conditions under which the coupling map $\mathbf{C}$ is such that

$$\tilde{\Pi}_m V = \mathbf{C}(\tilde{\Pi}_p Z) \tag{11}$$

In the Laplace domain, condition (P1) becomes $\bar{V}(s) = \bar{\mathbf{C}}(s)\bar{Z}(s)$, which, pre-multiplied by $\tilde{\Pi}_m$ becomes $\tilde{\Pi}_m \bar{V}(s) = \tilde{\Pi}_m \bar{\mathbf{C}}(s)\bar{Z}(s)$. Clearly, any linear map $\mathbf{C}$ with transfer function $\bar{\mathbf{C}}(s) \in \mathbb{C}^{Nm\times Np}$ satisfying (11) needs to also satisfy $\bar{\mathbf{C}}(s)\tilde{\Pi}_p = \tilde{\Pi}_m \bar{\mathbf{C}}(s)$. The following assumption and proposition give a condition under which (11) is satisfied.

Assumption 2: The interconnection between the N nodes is given by $W = -\tilde{\Gamma} V = -\tilde{\Gamma}\mathbf{C}(Z)$, where

- The map $\mathbf{C} : \mathcal{L}_{2e}^{Np} \to \mathcal{L}_{2e}^{Nm}$ satisfies (P1) and is a causal, locally Lipschitz continuous LTI system with transfer function
$$\bar{\mathbf{C}}(s) = \begin{bmatrix} \bar{C}_{1,1}(s) & \cdots & \bar{C}_{1,p}(s) \\ \vdots & \ddots & \vdots \\ \bar{C}_{m,1}(s) & \cdots & \bar{C}_{m,p}(s) \end{bmatrix} \tag{12}$$
where, for $k = 1, \cdots, m$, $l = 1, \cdots, p$, the map $\bar{C}_{k,l}(s) \in \mathbb{C}^{N\times N}$ satisfies $\bar{C}_{k,l}(s)\mathbf{1}_{N\times N} = \mathbf{1}_{N\times N}\bar{C}_{k,l}(s)$ and is such that $\bar{V}(s) = \bar{\mathbf{C}}(s)\bar{Z}(s)$.
- $\tilde{\Gamma}$ satisfies the coupling rule (P2)

Proposition 2: Under Assumption 2 the transfer function $\bar{\mathbf{C}}(s)$ is such that

$$\tilde{\Pi}_m \bar{W}(s) = -\tilde{\Gamma}\bar{\mathbf{C}}(s)\tilde{\Pi}_p \bar{Z}(s) \tag{13}$$

Proof: The transfer function of map $\mathbf{C}$ obeys $\bar{W}(s) = -\tilde{\Gamma}\bar{\mathbf{C}}(s)\bar{Z}(s)$. The rest of the proof follows from the fact that $\Pi\Gamma = \Gamma\Pi$ and that Assumption 2 implies the commutativity of transfer matrices $\bar{C}_{k,l}(s)$ with Π. ■

Remark 2: The class of transfer functions $\bar{C}_{k,l}(s)$ that satisfy the condition $\bar{C}_{k,l}(s)\mathbf{1}_{N\times N} = \mathbf{1}_{N\times N}\bar{C}_{k,l}(s)$ includes the class of circulant matrices and all binary permutations of the rows and columns of circulant matrices because the row and column sums of such matrices are equal.

Remark 3: Topologically, having a coupling matrix $\bar{\mathbf{C}}(s) = \{\bar{C}_{k,l}(s)\}$ that is non-diagonal means that nodal outputs $\mathbf{z}_j$ are combined from potentially several nodes to produce a signal $\mathbf{v}_j$ which is then distributed throughout the network using the Laplacian matrix $\tilde{\Gamma}$.

We now give a sufficient condition for the synchronization of dynamically coupled networks of nodes of the form (8).

Lemma 1: Consider a network of N nodes of the form (8). Suppose that Assumptions 1 and 2 are satisfied and suppose also that the map $\mathbf{G}$ in (8) is causal, locally Lipschitz continuous and incrementally finite-gain $\mathcal{L}_2$ stable as defined in Definition 4 with incremental $\mathcal{L}_2$ gain $\tilde{\gamma}(\mathbf{G})$ so that

$\|\tilde{\Pi}_n Y\| \le \tilde{\gamma}(\mathbf{G})\|\tilde{\Pi}_q R\| + \tilde{\eta}$. Defining the upper linear fractional transformation (LFT)

$$\mathcal{F}(\bar{\mathbf{H}}(s), \tilde{\Gamma}\bar{\mathbf{C}}(s)) = \\ \bar{\mathbf{H}}_{r_u}(s) - \bar{\mathbf{H}}_{r_w}(s)\tilde{\Gamma}\bar{\mathbf{C}}(s)\left(I_{Np} + \bar{\mathbf{H}}_{z_w}(s)\tilde{\Gamma}\bar{\mathbf{C}}(s)\right)^{-1}\bar{\mathbf{H}}_{z_u}(s) \tag{14}$$

then under the uniform continuity assumption of Theorem 1, the network synchronizes as $t \to \infty$ in the sense of Definition 6 if

$$\|\mathcal{F}(\bar{\mathbf{H}}(s), \tilde{\Gamma}\bar{\mathbf{C}}(s))\|_\infty < \frac{1}{\tilde{\gamma}(\mathbf{G})} \tag{15}$$

Proof: As in the proof of Theorem 1, the network interconnection composed of (8) and (4) constitutes a well-posed system as a result of the strong causality of $\mathbf{H}$.

From Assumptions 1 and 2, Propositions 1 and 2 show that the transfer functions $\bar{\mathbf{H}}(s)$ and $\bar{\mathbf{C}}(s)$ satisfy (10) and (13). Let $\bar{\mathbf{H}}(s) = \begin{bmatrix} \bar{\mathbf{H}}_z(s) \\ \bar{\mathbf{H}}_r(s) \end{bmatrix}$ where $\bar{\mathbf{H}}_z(s) = \begin{bmatrix} \bar{\mathbf{H}}_{z_w}(s) & \bar{\mathbf{H}}_{z_u}(s) \end{bmatrix}$ and $\bar{\mathbf{H}}_r(s) = \begin{bmatrix} \bar{\mathbf{H}}_{r_w}(s) & \bar{\mathbf{H}}_{r_u}(s) \end{bmatrix}$ By Assumption 1 and the map $Y = \mathbf{G}(R)$ we have

$$\begin{bmatrix} Z \\ Y \end{bmatrix} = \begin{bmatrix} \mathbf{H}_z([W^*, U^*]^*) \\ \mathbf{G}(\mathbf{H}_r([W^*, U^*]^*)) \end{bmatrix} = \mathbf{F}\left(\begin{bmatrix} W \\ U \end{bmatrix}\right) \tag{16}$$

where $\mathbf{H}_z$ and $\mathbf{H}_r$ are time domain maps the transfer functions of which are $\bar{\mathbf{H}}_z(s)$ and $\bar{\mathbf{H}}_r(s)$ respectively. By combining the maps $\mathbf{F}$ and $\mathbf{C}$ we arrive at the closed loop system $\tilde{\mathbf{F}}$ and hence we can apply Theorem 1 by verifying that the map $Y = \tilde{\mathbf{F}}(U)$ is incrementally finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain $\tilde{\gamma}(\tilde{\mathbf{F}}) < 1$. To find an upper bound on $\tilde{\gamma}(\tilde{\mathbf{F}})$, first consider the feedback system

$$\begin{bmatrix} \tilde{\Pi}_p\bar{Z}(s) \\ \tilde{\Pi}_q\bar{R}(s) \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{H}}_{z_w}(s) & \bar{\mathbf{H}}_{z_u}(s) \\ \bar{\mathbf{H}}_{r_w}(s) & \bar{\mathbf{H}}_{r_u}(s) \end{bmatrix} \begin{bmatrix} \tilde{\Pi}_m\bar{W}(s) \\ \tilde{\Pi}_n\bar{U}(s) \end{bmatrix} \\ \tilde{\Pi}_m\bar{W}(s) = -\tilde{\Gamma}\bar{\mathbf{C}}(s)\tilde{\Pi}_p\bar{Z}(s) \tag{17}$$

The map from $\tilde{\Pi}_n\bar{U}(s)$ to $\tilde{\Pi}_q\bar{R}(s)$ is given by $\tilde{\Pi}_q\bar{R} = \mathcal{F}(\bar{\mathbf{H}}(s), \tilde{\Gamma}\bar{\mathbf{C}}(s))\tilde{\Pi}_n U$ where $\mathcal{F}(\cdot,\cdot)$ is the upper LFT

$$\mathcal{F}(\bar{\mathbf{H}}(s), \tilde{\Gamma}\bar{\mathbf{C}}(s)) \\ = \bar{\mathbf{H}}_{r_u}(s) - \bar{\mathbf{H}}_{r_w}(s)\tilde{\Gamma}\bar{\mathbf{C}}(s)\left(I_{Np} + \bar{\mathbf{H}}_{z_w}(s)\tilde{\Gamma}\bar{\mathbf{C}}(s)\right)^{-1}\bar{\mathbf{H}}_{z_u}(s)$$

Note that the well-posedness property discussed earlier in the proof implies the invertibility of $\left(I_{Np} + \bar{\mathbf{H}}_{z_w}(s)\tilde{\Gamma}\bar{\mathbf{C}}(s)\right)$ (see, e.g. [21]). If $\mathcal{F}(\bar{\mathbf{H}}(s), \tilde{\Gamma}\bar{\mathbf{C}}(s)) \in \mathcal{H}_\infty$ its incremental $\mathcal{L}_2$ gain is then given by $\tilde{\gamma}\left(\mathcal{F}(\bar{\mathbf{H}}(s), \tilde{\Gamma}\bar{\mathbf{C}}(s))\right) = \|\mathcal{F}(\bar{\mathbf{H}}(s), \tilde{\Gamma}\bar{\mathbf{C}}(s))\|_\infty$. The map $\tilde{\mathbf{F}}$ is composed of the cascade of the maps $\mathcal{F}(\bar{\mathbf{H}}(s), \tilde{\Gamma}\bar{\mathbf{C}}(s))$ and $\mathbf{G}$, and their incremental $\mathcal{L}_2$ gains satisfy $\tilde{\gamma}(\tilde{\mathbf{F}}) \le \tilde{\gamma}(\mathcal{F}(\bar{\mathbf{H}}(s), \tilde{\Gamma}\bar{\mathbf{C}}(s)))\tilde{\gamma}(\mathbf{G})$. Therefore, by Theorem 1, the output synchronization of nodes of the form (8) in the sense of Definition 6 is achieved as $t \to \infty$ if $\|\mathcal{F}(\bar{\mathbf{H}}(s), \tilde{\Gamma}\bar{\mathbf{C}}(s))\|_\infty < \frac{1}{\tilde{\gamma}(\mathbf{G})}$. ∎

The following corollary of Lemma 1 and Theorem 1 gives sufficient conditions for the synchronization of network nodes in the case where the coupling operator $\mathbf{C}$ is block diagonal. This means that the output vector $\mathbf{z}_j$ from each node j is mapped onto the vector $\mathbf{v}_j$ via a transfer matrix in $\mathbb{C}^{m\times p}$ that we shall denote by $\check{C}(s)$ and which has the decomposition $\check{C}(s) = \{\check{C}_{k,l}(s)\}$ with $\check{C}_{k,l}(s) \in \mathbb{C}$. The signals v_{i_j}, $i = 1\cdots, m, j = 1, \cdots, N$ are then mapped onto the nodal input vector W via the concatenated Laplacian matrix $\tilde{\Gamma}$, as in (P2). The coupling operator $\mathbf{C}$ therefore takes the block form in (12), whereby $\bar{C}_{k,l}(s) = \check{C}_{k,l}(s)I_N$.

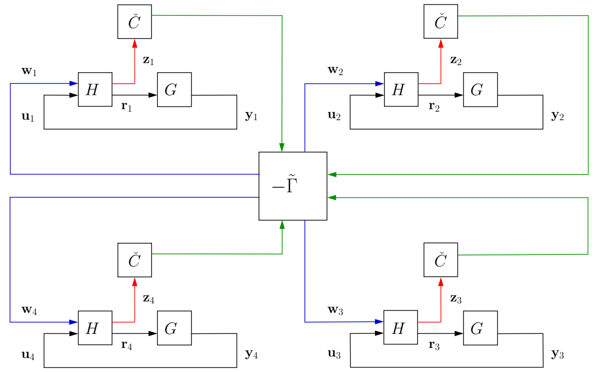


Fig. 5. Illustration for a network of four nodes of the form (8) with identical coupling blocks $\check{C}(s)$ and $\tilde{\Gamma} = I_m \otimes \Gamma$

Equivalently the transfer matrix of the coupling operator $\mathbf{C}$ also has the decomposition $\bar{\mathbf{C}}(s) = \check{C}(s) \otimes I_N$. This coupling topology is illustrated in Figure 5.

Corollary 1: Consider a network of N nodes (illustrated for $N = 4$ in Figure 5) of the form (8) satisfying the uniform continuity assumption of Theorem 1 in addition to Assumptions 1 and 2. Suppose that the map $\mathbf{G}$ in (8) is an incrementally finite-gain $\mathcal{L}_2$ stable map which has incremental $\mathcal{L}_2$ gain $\tilde{\gamma}(\mathbf{G})$. In the case where

- the Laplacian matrices $\Gamma_1 = \cdots = \Gamma_m = \Gamma$ are symmetric and have the diagonalization $\Gamma = \Phi\Lambda\Phi^*$, with $\Lambda = \mathrm{diag}\{\lambda_1, \cdots, \lambda_N\}$ and $\lambda_N \ge \lambda_{N-1} \ge, \cdots, \ge \lambda_2 > \lambda_1 = 0$.
- the transfer matrix $\bar{\mathbf{C}}(s)$ has the structure of (12) and for each k, l $\bar{C}_{k,l}(s) = \check{C}_{k,l}(s)I_N$, where $\check{C}(s) = \{\check{C}_{k,l}(s)\} \in \mathbb{C}^{m\times p}$ and $\check{C}_{k,l}(s) \in \mathbb{C}$.

the coupling transfer matrix $\bar{\mathbf{C}}(s) = \check{C}(s) \otimes I_N$ ensures synchronization of the nodes as $t \to \infty$ in the sense of Definition 6 if, for $j = 2, \cdots, N$,

$$\mathcal{F}(\bar{H}(s), \lambda_j\check{C}(s)) < \frac{1}{\tilde{\gamma}(\mathbf{G})} \tag{18}$$

where $\mathcal{F}(\bar{H}(s), \lambda_j\check{C}(s))$ is the upper LFT

$$\mathcal{F}(\bar{H}(s), \lambda_j\check{C}(s)) = \\ \bar{H}_{r_u}(s) - \lambda_j\bar{H}_{r_w}(s)\check{C}(s)[I_p + \lambda_j\bar{H}_{z_w}(s)\check{C}(s)]^{-1}\bar{H}_{z_u}(s)$$

Proof: Proof omitted due to lack of space. ∎

VI. EXAMPLES

In this section we will apply Corollary 1 to a network of nodes where each node is a dynamical system modelling the biochemical oscillator due to Goodwin [22]. We consider a network of $N = 6$ nodes of the form

$$\begin{aligned} \dot{x}_{1_j} &= -\tfrac{1}{5}x_{1_j} + u_{1_j} + w_{1_j}, \quad u_{1_j} = -y_{1_j} \\ \dot{x}_{2_j} &= -\tfrac{1}{5}x_{2_j} + \tfrac{1}{5}x_{1_j} \\ \dot{x}_{3_j} &= -\tfrac{1}{5}x_{3_j} + \tfrac{1}{5}x_{2_j} \\ z_{1_j} &= x_{3_j}, \quad r_{1_j} = x_{3_j}, \qquad y_{1_j} = g(r_{1_j}) = \begin{cases} -\frac{1}{1+r_{1_j}^{20}} & r_{1_j} \ge 0 \\ -1 & r_{1_j} < 0 \end{cases} \end{aligned} \tag{19}$$

We assume that the nodes are coupled using outputs $z_{1_j} = x_{3_j}$ and inputs w_{1_j} and we shall use Theorem 1 to find a coupling scheme $\mathbf{C}$ ensuring synchronization as $t \to \infty$ in the sense of Definition 6. As in (9) we have

$$\begin{bmatrix} \bar{Z}(s) \\ \bar{R}(s) \end{bmatrix} = \bar{\mathbf{H}}(s)\begin{bmatrix} \bar{W}(s) \\ \bar{U}(s) \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{H}}_{z_w}(s) & \bar{\mathbf{H}}_{z_u}(s) \\ \bar{\mathbf{H}}_{r_w}(s) & \bar{\mathbf{H}}_{r_u}(s) \end{bmatrix}\begin{bmatrix} \bar{W}(s) \\ \bar{U}(s) \end{bmatrix}$$

where $\bar{\mathbf{H}}_{z_w}(s) = \bar{\mathbf{H}}_{z_u}(s) = \bar{\mathbf{H}}_{r_w}(s) = \bar{\mathbf{H}}_{r_u}(s) = h(s) \otimes I_N$, and, from (19) $h(s) = \left(\begin{array}{c|c} A_h & B_h \\ \hline C_h & D_h \end{array}\right)$ where

$$A_h = \begin{bmatrix} -0.2 & 0 & 0 \\ 0.2 & -0.2 & 0 \\ 0 & 0.2 & -0.2 \end{bmatrix} \qquad B_h = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C_h = [\ 0 \quad 0 \quad 1\ ] \qquad\qquad D_h = 0$$

We also let $Y = \mathbf{G}(R) = [\ g(r_{1_1}) \quad \cdots \quad g(r_{1_6})\ ]^*$. As the map $\bar{\mathbf{H}}(s)$ satisfies Assumption 1, then by Proposition 1

$$\begin{bmatrix} \tilde{\Pi}_1 \bar{Z}(s) \\ \tilde{\Pi}_1 \bar{R}(s) \end{bmatrix} = \bar{\mathbf{H}}(s) \begin{bmatrix} \tilde{\Pi}_1 \bar{W}(s) \\ \tilde{\Pi}_1 \bar{U}(s) \end{bmatrix}$$

We assume that the nodal coupling satisfies Assumption 2: outputs $Z = Z_1 = [\ z_{1_1} \quad \cdots \quad z_{1_6}\ ]^*$ are input into an LTI system with transfer function $\bar{\mathbf{C}}(s) = \bar{C}_{1,1}(s) = c(s)I_N, c(s) \in \mathbb{C}$, which is of the form (12) and which represents the coupling dynamics. The output of $\bar{\mathbf{C}}(s)$ then diffuses through the network with Laplacian matrix $\tilde{\Gamma} = \Gamma_1$ and feeds into the nodal inputs w_{1_j}. Formally, we have $\bar{W}(s) = \bar{W}_1(s) = -\tilde{\Gamma}\bar{\mathbf{C}}(s)\bar{Z}(s) = -\Gamma_1 \bar{C}_{1,1}(s)\bar{Z}_1(s)$. This assumed coupling structure satisfies Assumption 2 as $\bar{C}_{1,1}(s)\mathbf{1}_{N\times N} = \mathbf{1}_{N\times N}\bar{C}_{1,1}(s)$, and therefore by Proposition 2, $\bar{\mathbf{C}}(s)$ is such that $\tilde{\Pi}_1 \bar{W}(s) = -\tilde{\Gamma}\bar{\mathbf{C}}(s)\tilde{\Pi}_1 \bar{Z}(s)$.

Now suppose that Γ_1 has the bidirectional ring structure

$$\Gamma_1 = \frac{\rho}{2} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \tag{20}$$

For this example, $\rho = 1$ and the coupling block is such that $\bar{\mathbf{C}}(s) = c(s)I_N$. The coupling matrix $\check{C}(s)$ in Corollary 1 is therefore given by $\check{C}(s) = c(s)$. The condition of Corollary 1 for synchronization is that

$$\begin{aligned} &\|\bar{H}_{r_u}(s) - \lambda \bar{H}_{r_w}(s)\check{C}(s)[I_p + \lambda \bar{H}_{z_w}(s)\check{C}(s)]^{-1}\bar{H}_{z_u}(s)\|_\infty \\ &= \left\| \frac{h(s)}{1+\lambda h(s)c(s)} \right\|_\infty < \frac{1}{\bar{\gamma}(\mathbf{G})} \end{aligned} \tag{21}$$

for $\lambda = \lambda_2, \cdots, \lambda_N$, the non-zero eigenvalues of Γ. Using the Matlab LMI Toolbox we can construct a coupling $c(s) = \left(\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right)$ satisfying (21) for $\lambda = \frac{\lambda_2 + \lambda_N}{2}$, where

$$\begin{aligned} A_r &= \begin{bmatrix} -0.2010 & -2.1295 & 13.6910 \\ 2.1295 & 2.6265 & -11.0119 \\ -13.6910 & -11.0119 & -8.1417 \end{bmatrix} \quad B_r = \begin{bmatrix} 2.0560 \\ 9.0191 \\ -68.2687 \end{bmatrix} \\ C_r &= [\ -2.5700 \quad 11.2739 \quad -85.3358\ ] \qquad D_r = 0 \end{aligned} \tag{22}$$

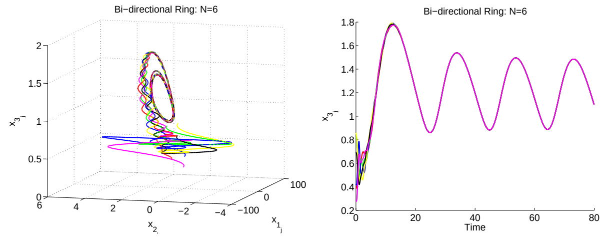
It can be verified that this coupling $c(s)$ also satisfies (21) for $j = 2, \cdots, N$ and therefore guarantees synchronization in the sense of Definition 6. Figure 6(a) shows the limit cycle of the states of $N = 6$ nodes of the form (19) interconnected using the coupling block $c(s)$ in (22) and Γ_1 with the bidirectional ring structure given above. Figure 6(b) shows the evolution with time of state x_{3_j}. Note that since y_{1_j} is a continuous function of x_{3_j} it follows that synchronization of output x_{3_j} implies synchronization of state y_{1_j}.

VII. Conclusion

A sufficient condition for the output synchronization of a network of dynamical systems has been presented. By considering the operator formed by combining the forward path of each node with the coupling, it was shown that if the coupling sufficiently reduces the incremental $\mathcal{L}_2$ gain of the operator then output synchronization is achieved. This led to a methodology for designing coupling architectures that ensure output synchronization.

The choice of possible synchronizing coupling input/output signals has been extended beyond [8], [12] by relaxing the condition that the network nodes be of relative degree one. The class of admissible coupling mechanisms has also been extended from linear, static maps to the class of $\mathcal{L}_{2e}$ operators.

Whilst the approaches to analyzing synchrony given in [8] and in this paper differ, these two studies share the same underlying idea of using the nodal coupling to decrease the incremental gains of the forward paths of network nodes in order to achieve incremental stability.



(a) Limit cycle (b) Time evolution of state x_{3_j}

Fig. 6. Synchronization of a $N = 6$ CFS network with a bi-directional ring topology.

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