Problem Sheet 1: Attracting/repelling fixed points in models of order 1

If you find any typos/errors in this problem sheet please email jk208@ic.ac.uk.

Consider the general order one model

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0 \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$. The point $\bar{x} \in \mathbb{R}$ is said to be a fixed point of (1) if $f(\bar{x}) = 0$. Let $\bar{x}$ be a fixed point of (1). Consider the set

$$\mathcal{A}_{\bar{x}} = \{y \in \mathbb{R} : x(0) = y \Rightarrow x(t) \to \bar{x} \text{ as } t \to +\infty\}$$

called the *region of attraction* of $\bar{x}$. It is the set of points such that if (1) is initialised on any of them, then the solution of (1), $x(t)$, converges to $\bar{x}$. If there exists an open interval around $\bar{x}$, that is, a set

$$B_\delta(\bar{x}) = \{y \in \mathbb{R} : |\bar{x} - y| < \delta\}$$

for some $\delta > 0$, that is contained in the region of attraction of $\bar{x}$, i.e., $B_\delta(\bar{x}) \subset \mathcal{A}_{\bar{x}}$, then $\bar{x}$ is said to be *locally attracting*. If all intervals around $\bar{x}$ are contained in the region of attraction, that is, $B_\delta(\bar{x}) \subset \mathcal{A}_{\bar{x}}$ for any $\delta > 0$, then $\bar{x}$ is said to be *globally attracting*. Notice that this is equivalent to the region of attraction being the set of all real numbers, i.e., $\mathcal{A}_{\bar{x}} = \mathbb{R}$. In addition, notice that if a fixed point is globally attracting it is also locally attracting (but not necessarily vice versa!). Finally, we say that the fixed point is *repelling* if it is not locally attracting.

1. For the following models sketch the phase line ($\dot{x}$ vs x). Draw on the x-axis arrows indicating the direction of the flow, that is whether x is increasing or decreasing in time. Use your sketch to find all of the fixed points and their respective regions of attraction. Using the regions of attraction classify the fixed points as locally attracting, globally attracting or repelling.
   (a) $\dot{x} = \tan^{-1}(x)$.
   (b) $\dot{x} = x^2 - 4$.
   (c) $\dot{x} = x - x^3$.
   (d) $\dot{x} = sin(x)$.
   (e) $\dot{x} = |x|$.
   (f) $\dot{x} = -x^3 + (\alpha + \beta)x^2 - \alpha\beta x, \quad$ where $0 < \alpha < \beta$ are constants.

Solutions

1. (a) The phase line should look like

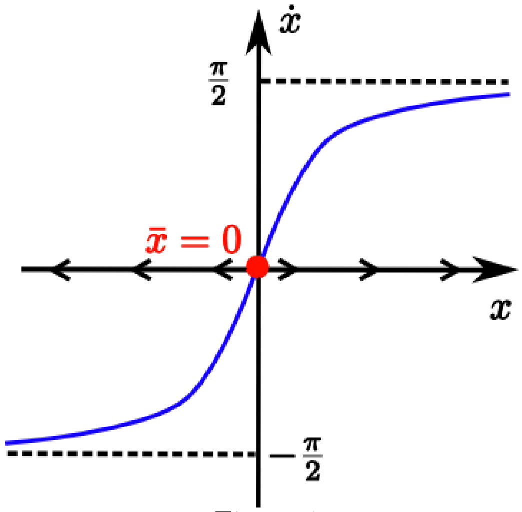

Figure 1

There is a unique fixed point given by

$$\dot{x} = 0 \Leftrightarrow \tan^{-1}(\bar{x}) = 0 \Leftrightarrow \bar{x} = tan(0) \Leftrightarrow \bar{x} = 0.$$

Notice that if $x > 0$ then $\dot{x} > 0$ hence x will only increase as time passes. Similarly, if $x < 0$ then $\dot{x} < 0$ hence x will only decrease as time passes. Thus, if $x_0 > 0$ or $x_0 < 0$ then $x(t)$ will not tend to the fixed point 0 as time passes. So the region of attraction is given by

$$\mathcal{A}_0 = \{0\}.$$

The region of attraction of 0 contains only 0 itself. Hence, we cannot find any $\delta > 0$ such that the open interval $B_\delta(0) = (-\delta, \delta)$ is contained in $\mathcal{A}_0$, in other words, such that $B_\delta(0) \subset \mathcal{A}_0$. More explicitly, for any $\delta > 0$, $-\delta < \delta/2 < \delta$ hence $\delta/2 \in B_\delta(0)$ and $\delta/2 > 0$ hence $\delta/2 \notin \mathcal{A}_0$. So $B_\delta(0) \not\subset \mathcal{A}_0$. Hence, 0 is not locally attractive and thus it is repelling.

(b) The phase line should look like

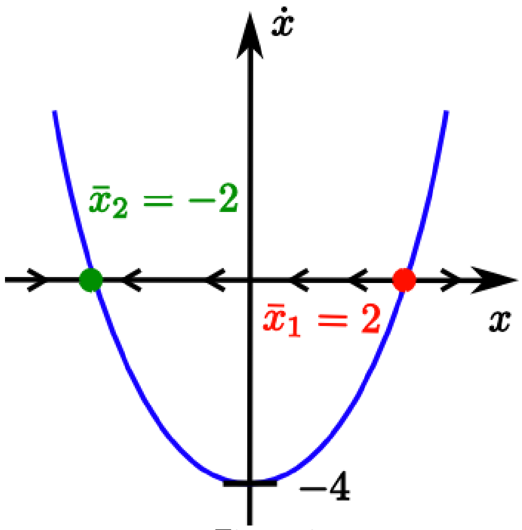

Figure 2

It's a quadratic, thus it has two roots given by

$$\dot{x} = 0 \Leftrightarrow \bar{x}^2 = 4 \Rightarrow \bar{x}_1 = 2, \bar{x}_2 = -2.$$

Using the flow directions we see that,

$$\mathcal{A}_{-2} = (-\infty, 2), \qquad \mathcal{A}_2 = \{2\}.$$

As in part (a) we can conclude that 2 is a repelling fixed point. Consider the other fixed point, -2, and note that $(-2-\delta, -2+\delta) \subset \mathcal{A}_{-2}$ for any $0 < \delta \leq 4$. Thus -2 is locally attracting. However, it is not globally attracting because $\mathcal{A}_{-2} \neq \mathbb{R}$. Notice that the fact that there exists another fixed point, 2, regardless whether 2 is attracting or repelling, implies that -2 cannot be globally attracting ($x_0 = 2 \Rightarrow x(t) = 2$ for all $t \geq 0 \Rightarrow x(t) \not\to -2$ as $t \to +\infty \Rightarrow 2 \notin \mathcal{A}_{-2} \Rightarrow \mathcal{A}_{-2} \neq \mathbb{R} \Rightarrow -2$ is not globally attracting).

(c) The phase line should look like

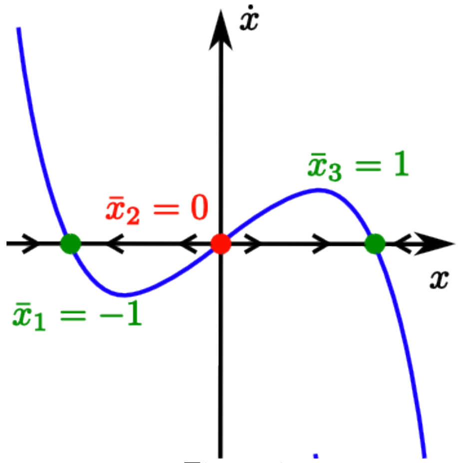

Figure 3

It's a cubic, thus it has three roots given by

$$\dot{x} = 0 \Leftrightarrow \bar{x} - \bar{x}^3 = (1 - \bar{x}^2)\bar{x} = 0 \Rightarrow \bar{x}_1 = -1, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 1.$$

Using the flow directions we see that,

$$\mathcal{A}_{-1} = (-\infty, 0), \qquad \mathcal{A}_0 = \{0\}, \qquad \mathcal{A}_1 = (0, +\infty).$$

Hence following similar logic as in parts (a) and (b), -1 and 1 are locally attracting and 0 is repelling.

(d) The phase line should look like

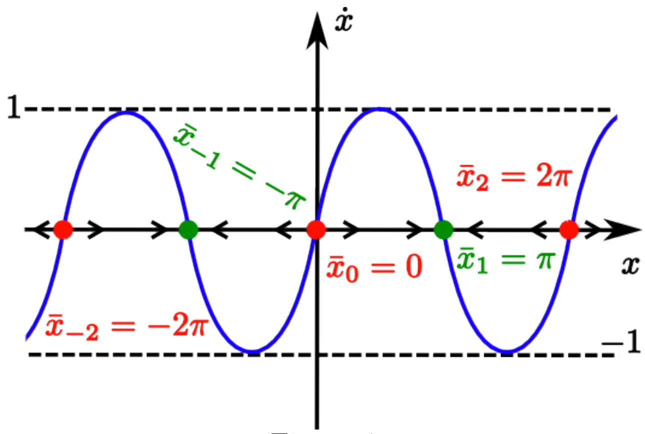

Figure 4

It's a sinusoid, thus it has infinitely many roots given by

$$\dot{x} = 0 \Leftrightarrow sin(\bar{x}) = 0 \Rightarrow \bar{x}_k = k\pi, \quad k \in \mathbb{Z}$$

where $\mathbb{Z}$ denotes the set of integers. Using the flow directions we see that for any even integer m and for any odd integer n

$$\mathcal{A}_{m\pi} = \{m\pi\}, \qquad \mathcal{A}_{n\pi} = ((n-1)\pi, (n+1)\pi).$$

Thus for all the even integers m the fixed points $m\pi$ are repelling and for all the odd integers n the fixed points $n\pi$ are locally attracting.

(e) The phase line should look like

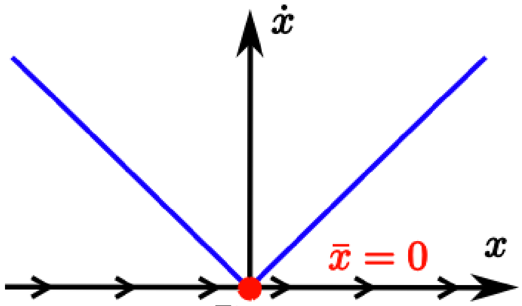

Figure 5

The absolute value of a number is zero if and only if that number is zero, thus

$$\dot{x} = 0 \Leftrightarrow |\bar{x}| = 0 \Leftrightarrow \bar{x} = 0.$$

Using the flow directions we see that

$$\mathcal{A}_0 = (-\infty, 0].$$

Even though 0 has a large region of attraction (more than half the real numbers!), 0 is at the boundary of that region. Hence, we are unable to find an open interval *centred around* 0 that is contained in the region of attraction of 0. In other words for any $\delta > 0$, $\delta/2 \notin (-\infty, 0] = \mathcal{A}_0$. Therefore, $\delta/2 \in (-\delta, \delta) \Rightarrow (-\delta, \delta) \not\subset \mathcal{A}_0$. Hence, 0 is a repelling fixed point.

(f) Factorising $\dot{x}$ makes our life considerably easier. Indeed, this phase line is quite similar to the one in part (c). So, keeping in mind that $0 < \alpha < \beta$.

$$\dot{x} = -x^3 + (\alpha+\beta)x^2 - \alpha\beta x = -x(x^2 - (\alpha+\beta)x + \alpha\beta) = -x(x-\alpha)(x-\beta). \tag{2}$$

Hence, $\dot{x}$ is a cubic that crosses the x-axis three times (at the fixed points $\bar{x}_1 = 0$, $\bar{x}_2 = \alpha$ and $\bar{x}_3 = \beta$). In addition, from (2), we have that: $x \in (-\infty, 0) \Rightarrow \dot{x} > 0$; $x \in (0, \alpha) \Rightarrow \dot{x} < 0$; $x \in (\alpha, \beta) \Rightarrow \dot{x} > 0$; and $x \in (\beta, +\infty) \Rightarrow \dot{x} < 0$. Using this information, we know that the phase line should look like in Figure 6.

Then, using the flow directions we see that

$$\mathcal{A}_0 = (-\infty, \alpha), \quad \mathcal{A}_\alpha = \{\alpha\}, \quad \mathcal{A}_\beta = (\alpha, +\infty).$$

Thus 0 and β are locally attracting while α is repelling.

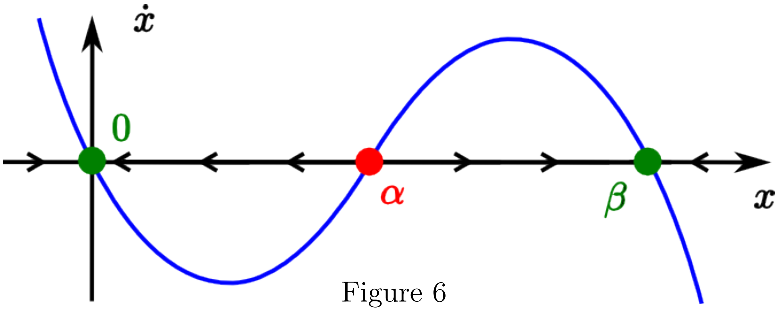

Figure 6