Problem Sheet 4: Second order non-linear models

If you find any typos/errors in this problem sheet please email jk208@ic.ac.uk.

1. Consider the model of arbitrary order n

$$\dot{x}(t) = f(x(t)) \tag{1}$$

where $x(t) \in \mathbb{R}^n$. Suppose that (1) has a fixed point $p \in \mathbb{R}^n$, i.e., p satisfies $f(p) = 0$, and let $J(p)$ denote the Jacobian of (1) evaluated at this fixed point p. Then, the Hartman-Grobman Theorem states that if no eigenvalue of $J(p)$ has a real part equal to zero[1] then p *locally behaves* like the fixed point at the origin of the *linearised model*[2]

$$\dot{z}(t) = J(p)z(t) \tag{2}$$

where $z \in \mathbb{R}^n$. By this we mean that there exists a ball centred around p of sufficiently small radius (possibly very small) in which the phase portrait of the fixed point p of (1) "looks arbitrarily close" to that of the fixed point at the origin of (2). That is, if the origin of (2) is an unstable node, then p of (1) acts as an unstable node in that ball (with the same eigenvectors/eigenvalues), if the origin of (2) is an unstable spiral, then $\ldots$, etc.

Let's illustrate the Hartman-Grobman Theorem by applying it to the following second order model

$$\dot{x} = 2\mu(1 - y^2)x - atan(y) \tag{3a}$$
$$\dot{y} = x \tag{3b}$$

where $\mu \in \mathbb{R}$ is a parameter.

(a) Find all fixed points of (3).

(b) Compute the analytical expression of the Jacobian matrix of (3).

(c) For each fixed point p, what can you deduce about p using the Hartman-Grobman Theorem? (consider all possible values of $\mu \in \mathbb{R}$)

2. Consider the second order model[3]

$$\dot{x} = x + y - x^3 \tag{4a}$$
$$\dot{y} = y - x - y^3. \tag{4b}$$

(a) Find all fixed points of (4). *Hint: Plot the nullclines and look for intersections.*

(b) Find and evaluate the Jacobian at each fixed point. For each fixed point p, what can you deduce about p using the Hartman-Grobman Theorem discussed in exercise 1?

(c) Show that any ball centred around the origin and of sufficiently large radius R, that is any set $\mathcal{B}_R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$ where R is "sufficiently large" (i.e., is larger than a certain quantity to be computed), is forward invariant.

(d) Using your answers to parts (a) - (c) and the Poincaré-Bendixson Theorem, what can you say about the existence of a limit cycle?

[1] If the Jacobian matrix $J(p)$ is such that none of its eigenvalues has a zero real part, we say that p is a *hyperbolic* fixed point.

[2] We sometimes refer to the process of finding this linearised model as "linearising around the fixed point p".

[3] This question was provided by Dr Angeli and it's taken from his course EE4-23 Stability and Control of Non-linear Systems.

3. In 1979, Schnackenberg suggested that the following three biochemical reactions could lead to oscillations if the concentrations of species A and B were fixed at a constant level.

$$X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} A, \quad 2X + Y \xrightarrow{k_2} 3X, \quad B \xrightarrow{k_3} Y$$

Assuming that the concentrations of A and B are constant and applying the law of mass action one can model the above system of reactions as

$$\frac{d[X]}{dt} = k_{-1}[A] - k_1[X] + k_2[X]^2[Y] \tag{5a}$$

$$\frac{d[Y]}{dt} = k_3[B] - k_2[X]^2[Y] \tag{5b}$$

where $[A] > 0$, $[B] > 0$, $[X] > 0$ and $[Y] > 0$ denote the concentrations of A, B, X and Y, respectively; $k_1 > 0$ and $k_{-1} > 0$ denote the forward and backward reaction coefficients of the first reaction, respectively; and $k_2 > 0$ and $k_3 > 0$ denote the reaction coefficients of the second and third reactions, respectively.

(a) Use the substitutions $X = \alpha x_1$, $Y = \beta x_2$ and $t = \xi\tau$ to non-dimensionalise model (5) and obtain the non-dimensionalised model

$$\frac{dx_1}{d\tau} = \sigma - \gamma x_1 + x_1^2 x_2 \tag{6a}$$

$$\frac{dx_2}{d\tau} = 1 - x_1^2 x_2 \tag{6b}$$

where $\sigma > 0$ and $\gamma > 0$ are two new parameters. Give expressions of σ and γ in terms of the original parameters $[A]$, $[B]$, k_1, k_{-1}, k_2 and k_3.

(b) Find all fixed points of the non-dimensionalised model (6).

(c) Consider the trapezoidal region in the phase plane shown in the figure below. Do the fixed point/s lie inside the trapezoidal region? Show that this trapezoidal region is a trapping region (i.e., is forward invariant) for the model (6).

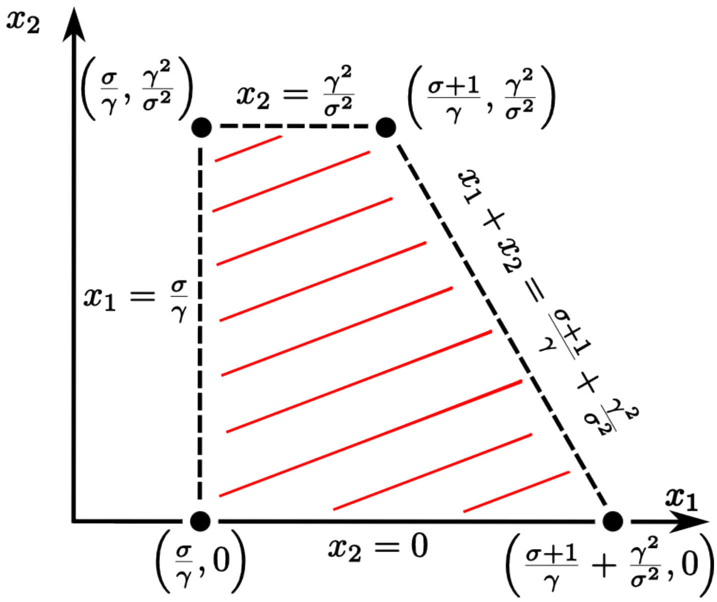

Figure 1: Candidate trapping region (shaded in red).

(d) Compute the Jacobian matrix, $J(x)$, of the non-dimensionalised model (6). **Do not** plug in the expressions for the coordinate of the fixed point/s you obtained in the part (b).

(e) Suppose that $\sigma = 5$ and $\gamma = 6$. Use your answers to parts (b) and (d) and sketch the local phase portrait of each fixed point. Classify each fixed point as either a stable/unstable node or a stable/unstable spiral. *Hint: This requires you to work out the eigenvalues (and if the fixed point in question is a node, the eigenvectors as well) of the Jacobian matrix.*

(f) Repeat part (e) but this time using $\sigma = \frac{1}{2}$ and $\gamma = 3$.

(g) Using your answers to parts (c), (e), (f) and the Poincaré-Bendixson Theorem, what can you say about the existence of a limit cycle if i) $\sigma = 5$ and $\gamma = 6$, ii) $\sigma = \frac{1}{2}$ and $\gamma = 3$.

Solutions

1. (a) At any fixed point $p = (\bar{x}, \bar{y})$, we have $\dot{x} = \dot{y} = 0$.
   Let's start with $\dot{y} = 0$: $\dot{y} = 0 \Leftrightarrow \bar{x} = 0$
   Next, $\dot{x} = 0 \Leftrightarrow 2\mu(1-\bar{y}^2)\bar{x} - atan\,(\bar{y}) = 2\mu(1-\bar{y}^2)(0) - atan\,(\bar{y}) = 0 \Leftrightarrow atan\,(\bar{y}) = 0 \Leftrightarrow \bar{y} = 0$.
   Thus, (3) has a unique fixed point $p = (0, 0)$.

   (b)
$$J(x,y) = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{bmatrix} = \begin{bmatrix} 2\mu(1-y^2) & -(4\mu xy + \frac{1}{1+y^2}) \\ 1 & 0 \end{bmatrix}$$
   .

   (c) The Jacobian evaluated at the origin gives
$$J(0,0) = \begin{bmatrix} 2\mu & -1 \\ 1 & 0 \end{bmatrix}$$
   The eigenvalues are the roots of $\det(J(0,0) - \lambda I) = \lambda^2 - 2\mu\lambda + 1$. Hence, $\lambda_{1,2} = \mu \pm \sqrt{\mu^2 - 1}$. Hence applying the Hartman-Grobman Theorem we can conclude that
   - If $0 < \mu < 1$, then the local phase portrait p resembles that of an unstable spiral.
   - If $\mu \geq 1$, then the local phase portrait p resembles that of an unstable node (note that $\mu^2 > \mu^2 - 1 \geq 0 \Rightarrow \mu > \sqrt{\mu^2 - 1} \Rightarrow \mu - \sqrt{\mu^2 - 1} > 0$).
   - If $\mu = 0$, both the eigenvalues are $\pm i$, thus the Hartman-Grobman Theorem is not applicable and one cannot deduce the local stability properties of (1) from the stability properties of the linearised system (2).[4]
   - If $-1 < \mu < 0$, then the local phase portrait of p resembles that of a stable spiral.
   - If $\mu \leq -1$, then the local phase portrait p resembles that of a stable node.

2. (a) If one attempts to solve for the fixed points algebraically, one ends up trying to find the roots of an 8^{th} order polynomial - which is far from desirable. In these sorts of situations, plotting the nullclines and checking graphically for intersections (which give the fixed points) will, sometimes, solve the problem. So, following the hint we have the x nullcline $\mathcal{N}_x = \{(x, y) : y = x^3 - x\}$ and the y nullcline $\mathcal{N}_y = \{(x, y) : x = y - y^3\}$, see Figure 2.

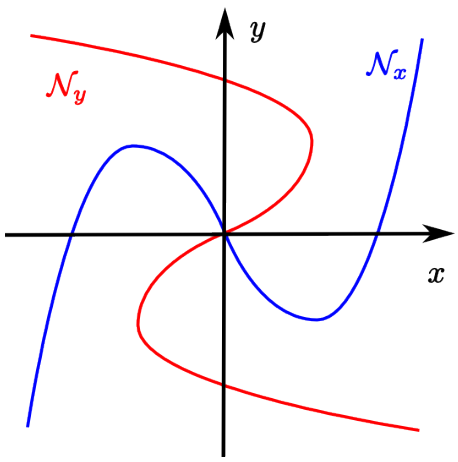

Figure 2: Nullclines.

[4] In such a case, the local stability property of the fixed point of (1) needs to be established by explicitly taking into account the nonlinear terms, e.g., using Lyapunov's direct method (constructing a Lyapunov function) or the Centre Manifold Theorem (these methods are beyond the scope of this course and will not be covered here).

From the figure, it is fairly clear that the nullclines only intersect at the origin. Thus $p = (0, 0)$ is the unique fixed point. This said, one could play devil's advocate and ask how can we be certain that something along the lines of what is plotted in Figure 3 is not the actual situation and we really have more than one fixed point (which would be a perfectly valid question). The short answer is because of both symmetry and that the coefficients in the cubics defining the nullclines are not large enough (the curve would have to be quite steep, compare Figures 2 and 3). However if one wanted to check carefully, for example, that the nullclines do not cross in the second quadrant, one could work out the coordinates of the local maximum (q_1, q_2) of $\mathcal{N}_x$, and then check that $(q_1, \alpha) \notin \mathcal{N}_y$ where $0 \leq \alpha \leq q_2$ (a good exercise is to actually prove that the nullclines don't intersect, try it).

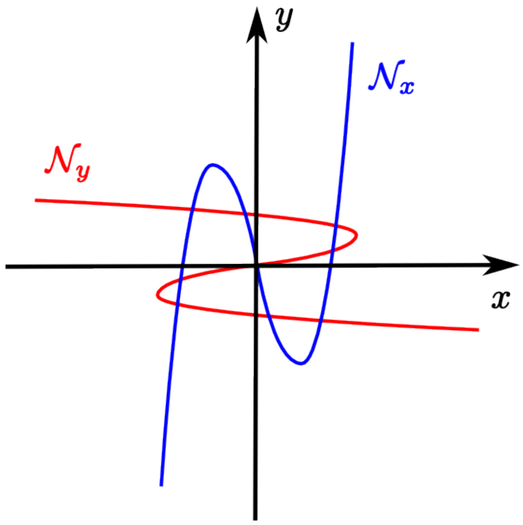

Figure 3: Are there nine fixed points?

(b)

$$J(x, y) = \begin{bmatrix} 1 - 3x^2 & 1 \\ -1 & 1 - 3y^2 \end{bmatrix} \Rightarrow J(0, 0) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

The eigenvalues are the roots of the characteristic polynomial $\det(J(0, 0) - \lambda I) = (\lambda - 1)^2 + 1 = \lambda^2 - 2\lambda + 2 \Rightarrow \lambda_{1,2} = 1 \pm i$. Neither eigenvalue has zero real part, thus, by the Hartman-Grobman Theorem, the fixed point at the origin of (4) behaves locally as an unstable spiral (rotating in the clockwise direction, can you see why? *Hint: for example, look at the sign of $\dot{y}$ on the x-axis (i.e., when $y = 0$). You will see that, on the x-axis, you have $\dot{x} = x - x^3$ and $\dot{y} = -x$, therefore the vector field on the x-axis is such that trajectories are rotating clockwise).*

(c) We have that $\mathcal{B} = \{(x, y) \in \mathbb{R}^2 : g(x) \leq 0\}$ where $g(x) = x^2 + y^2 - R^2$. Note that $g(x)$ is continuously differentiable. Thus, $\mathcal{B}$ is forward invariant if and only if, for all (x, y) such that $g(x, y) = 0$, i.e., for all (x, y) such that $x^2 + y^2 = R^2$, we have

$$\frac{dg(x(t), y(t))}{dt} = \begin{bmatrix} \frac{\partial g(x,y)}{\partial x} & \frac{\partial g(x,y)}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \end{bmatrix} = \begin{bmatrix} 2x & 2y \end{bmatrix} \begin{bmatrix} x + y - x^3 \\ y - x - y^3 \end{bmatrix} = 2(x^2 - x^4 + y^2 - y^4)$$

$= 2\left((x^2 + y^2) - (x^4 + y^4)\right) \leq 0.$

Hence, suppose that $x^2 + y^2 = R^2$, then, $\frac{dg(x(t),y(t))}{dt} = 2(R^2 - (x^4 + y^4))$. In addition, notice that because x^2 and y^2 are non-negative and $x^2 + y^2 = R^2$, then either $x^2 \geq \frac{R^2}{2}$ or $y^2 \geq \frac{R^2}{2}$ (or $x^2 = y^2 = \frac{R^2}{2}$). Hence, $x^4 + y^4 \geq \left(\frac{R^2}{2}\right)^2 = \frac{R^4}{4}$. So, $2(R^2 - (x^4 + y^4)) \leq 2(R^2 - \frac{R^4}{4}) = 2R^2(1 - \frac{R^2}{4}) \leq 0$ for any $R \geq 2$.

Hence, any ball centred around the origin with radius greater or equal than 2 is forward invariant.

(d) In part (c) we showed that the first condition of the Poincaré-Bendixson Theorem is satisfied, i.e., that there exists a bounded forward invariant subset of the phase plane. More specifically, we showed that all balls $\mathcal{B}$ centred at the origin and with radius larger or equal to 2 are forward invariant. From part (a) we know that there is a fixed point inside $\mathcal{B}$, namely the origin. Thus the second condition of the theorem is not satisfied. Fortunately, by linearising around the origin in part (b) we verified that the origin is an unstable spiral, thus, as discussed in the notes, we can safely assume that this locally unstable fixed point (i.e., the origin in this case) can be "cut out" of the region $\mathcal{B}$ without any further analysis on our part. Thus, we can conclude that there exists at least a limit cycle inside the forward invariant set $\mathcal{B}$ and that all trajectories that enter $\mathcal{B}$ eventually converge to a limit cycle.

3. (a) Plugging in the substitutions and moving the constants to the right hand side of both equations one obtains

$$\frac{dx_1}{d\tau} = \frac{\xi k_{-1}[A]}{\alpha} - \xi k_1 x_1 + \xi\alpha\beta k_2 x_1^2 x_2$$
$$\frac{dx_2}{d\tau} = \frac{\xi k_3[B]}{\beta} - \xi\alpha^2 k_2 x_1^2 x_2.$$

Next, we need to find values of α, β and ξ such that $\frac{\xi k_3[B]}{\beta} = 1$, $\xi\alpha^2 k_2 = 1$ and $\xi\alpha\beta k_2 = 1$.

First, $\frac{\xi k_3[B]}{\beta} = 1 \Rightarrow \beta = \xi k_3[B]$. Next, $\xi\alpha\beta k_2 = \xi^2\alpha[B]k_2k_3 = 1 \Rightarrow \alpha = \frac{1}{[B]k_2k_3\xi^2}$. Lastly, $\xi\alpha^2 k_2 = \frac{1}{[B]^2k_2k_3^2\xi^3} = 1 \Rightarrow \xi = \sqrt[3]{\frac{1}{[B]^2k_2k_3^2}}$. Now that ξ is expressed solely in terms of the parameters of (5), we can find the expression of α and β in terms of these parameters by substituting in the final expression of ξ.

Then we finish by setting $\sigma = \frac{\xi k_{-1}[A]}{\alpha} = \xi^3 k_{-1}[A][B]k_2k_3 = \frac{k_{-1}[A]}{k_3[B]}$ and $\gamma = k_1\xi = k_1\sqrt[3]{\frac{1}{[B]^2k_2k_3^2}}$.

(b) At the fixed point $p = (\bar{x}_1, \bar{x}_2)$, we have that $\frac{dx_1}{d\tau}|_{(\bar{x}_1,\bar{x}_2)} = \frac{dx_2}{d\tau}|_{(\bar{x}_1,\bar{x}_2)} = 0$.
First, $\frac{dx_2}{d\tau}|_{(\bar{x}_1,\bar{x}_2)} = 0 \Rightarrow 1 - \bar{x}_1^2\bar{x}_2 = 0 \Rightarrow \bar{x}_1^2\bar{x}_2 = 1$. Then, $\frac{dx_1}{d\tau}|_{(\bar{x}_1,\bar{x}_2)} = 0 \Rightarrow \sigma - \gamma\bar{x}_1 + \bar{x}_1^2\bar{x}_2 = \sigma - \gamma\bar{x}_1 + 1 = 0 \Rightarrow \bar{x}_1 = \frac{\sigma+1}{\gamma}$. Lastly, using $1 - \bar{x}_1^2\bar{x}_2 = 0$ again we get $\bar{x}_2 = \frac{1}{\bar{x}_1^2} = \frac{\gamma^2}{(1+\sigma)^2}$.

Hence we conclude there is a unique fixed point $p = \left(\frac{\sigma+1}{\gamma}, \frac{\gamma^2}{(1+\sigma)^2}\right)$.

(c) From part (b) we have that the unique fixed point of (6) is $p = (p_1, p_2) = \left(\frac{\sigma+1}{\gamma}, \frac{\gamma^2}{(1+\sigma)^2}\right)$. Examining the figure one can see that it lies inside the trapezoidal region if and only if $0 \leq p_2 \leq \frac{\gamma^2}{\sigma^2}$. But $\gamma > 0, \sigma > 0 \Rightarrow \frac{\gamma^2}{(1+\sigma)^2} > 0$ and $\sigma > 0 \Rightarrow (1+\sigma)^2 > \sigma^2 \Rightarrow \frac{\gamma^2}{(1+\sigma)^2} < \frac{\gamma^2}{\sigma^2}$. So yes, the fixed point does lie inside the trapezoidal region.

One can show that the trapezoidal region is forward invariant by showing that the vector field $f(x)$ (given by (6)) evaluated along the boundary of the trapezoidal region always points "inside" the trapezoidal region. This is so if along each of the four line segments that define the boundary of the region, the angle between $f(x)$ and the normal vector to the line segment (pointing outwards of the trapezoidal region), n, is greater or equal to $\pi/2$ (i.e., $n^T f(x) \leq 0$).

As shown in the figure the four line segments defining the trapezoidal region boundary are given by $l_1 = \{(x_1, 0) : \frac{\sigma}{\gamma} \leq x_1 \leq \frac{\sigma+1}{\gamma} + \frac{\gamma^2}{\sigma^2}\}$, $l_2 = \{(x_1, \frac{\sigma+1}{\gamma} + \frac{\gamma^2}{\sigma^2} - x_1) : \frac{\sigma+1}{\gamma} \leq x_1 \leq \frac{\sigma+1}{\gamma} + \frac{\gamma^2}{\sigma^2}\}$, $l_3 = \{(x_1, \frac{\gamma^2}{\sigma^2}) : \frac{\sigma}{\gamma} \leq x_1 \leq \frac{\sigma+1}{\gamma}\}$ and $l_4 = \{(\frac{\sigma}{\gamma}, x_2) : 0 \leq x_2 \leq \frac{\gamma^2}{\sigma^2}\}$.

The normal vectors then are $n_1 = (0,-1)^T$, $n_2 = (1,1)^T$, $n_3 = (0,1)^T$ and $n_4 = (-1,0)^T$, respectively. In what follows, we denote $fracdx_1d\tau$ by $\dot{x}_1$ and $fracdx_2d\tau$ by $\dot{x}_2$. The dot products are thus:

$$x \in l_1 \Rightarrow n_1^T f(x)|_{x\in l_1} = -\dot{x}_2|_{x\in l_1} = -(1 - x_1^2 x_2)|_{x\in l_1} = -1 \leq 0,$$

$$x \in l_2 \Rightarrow n_2^T f(x)|_{x\in l_2} = \dot{x}_1 + \dot{x}_2|_{x\in l_2} = (\sigma + 1 - \gamma x_1)|_{x\in l_2} \leq \sigma + 1 - \gamma\frac{\sigma+1}{\gamma} = 0,$$

$$x \in l_3 \Rightarrow n_3^T f(x)|_{x\in l_3} = \dot{x}_2|_{x\in l_3} = (1 - x_1^2 x_2)|_{x\in l_3} \leq 1 - \left(\frac{\sigma}{\gamma}\right)^2 \frac{\gamma^2}{\sigma^2} = 0,$$

$$x \in l_4 \Rightarrow n_4^T f(x)|_{x\in l_4} = -\dot{x}_1|_{x\in l_4} = -(\sigma - \gamma x_1 + x_1^2 x_2)|_{x\in l_4} \leq \gamma x_1 - \sigma|_{x\in l_4} = \gamma\frac{\sigma}{\gamma} - \sigma = 0$$

Thus, we can conclude that the trapezoidal region is forward invariant with respect to (6).

(d) $J(x) = \begin{bmatrix} 2x_1x_2 - \gamma & x_1^2 \\ -2x_1x_2 & -x_1^2 \end{bmatrix}$

(e) Plugging in $\sigma = 5$ and $\gamma = 6$ we have that the fixed point $p = \left(\frac{\sigma+1}{\gamma}, \frac{\gamma^2}{(1+\sigma)^2}\right) = (1,1)$. Next, $J(p) = \begin{bmatrix} 2x_1x_2 - \gamma & x_1^2 \\ -2x_1x_2 & -x_1^2 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ -2 & -1 \end{bmatrix}$.
The eigenvalues are the roots of the characteristic polynomial $\det(J(p) - \lambda I) = (\lambda+4)(\lambda+1) + 2 = \lambda^2 + 5\lambda + 6 = (\lambda+3)(\lambda+2)$. Hence the two eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -2$. The corresponding eigenvectors are given by the non-trivial solutions to $J(p)v_i = \lambda_i v_i$. So, $\begin{bmatrix} -4 & 1 \\ -2 & -1 \end{bmatrix} v_1 = -3v_1 \Rightarrow v_1 = [1,1]^T$ and $\begin{bmatrix} -4 & 1 \\ -2 & -1 \end{bmatrix} v_2 = -2v_2 \Rightarrow v_2 = [1/2,1]^T$.
From the above we can conclude that p is a stable node with fast eigenvector $[1,1]^T$ and slow eigenvector $[1/2,1]^T$. Thus, the phase portrait should look something similar to:

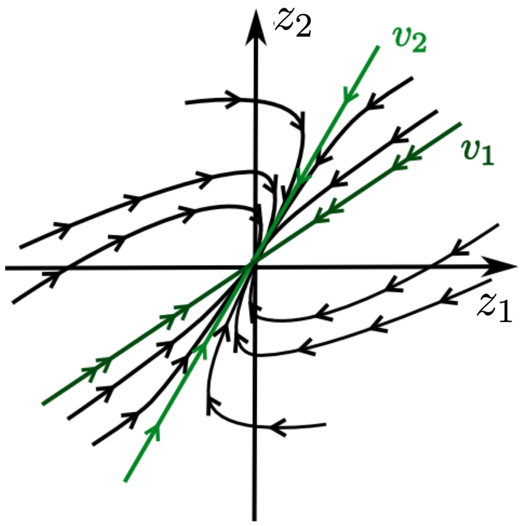

Figure 4: Phase portrait.

(f) Plugging in $\sigma = \frac{1}{2}$ and $\gamma = 3$ we have that the fixed point $p = \left(\frac{\sigma+1}{\gamma}, \frac{\gamma^2}{(1+\sigma)^2}\right) = (1/2,4)$. Next, $J(p) = \begin{bmatrix} 2x_1x_2 - \gamma & x_1^2 \\ -2x_1x_2 & -x_1^2 \end{bmatrix} = \begin{bmatrix} 1 & 1/4 \\ -4 & -1/4 \end{bmatrix}$.
The eigenvalues are the roots of the characteristic polynomial $\det(J(p) - \lambda I) = (\lambda-1)(\lambda+\frac{1}{4}) + 1 = \lambda^2 - \frac{3}{4}\lambda + \frac{3}{4}$. Hence the two eigenvalues are $\lambda_{1,2} = \frac{3}{8} \pm \sqrt{-\frac{39}{64}}$.

From the above we can conclude that p is an unstable spiral. To tell whether the trajectories will spin in the clockwise or anticlockwise direction we can simply examine the vector field the linearised model $\dot{z} = J(p)z$ at a single point $z \neq 0$. $J(p)[1,0]^T = [1,-4]^T$ which is in the 4^{th} quadrant, hence the trajectories are spinning in the clockwise direction. Thus, the phase portrait should look something similar that in Figure 5.

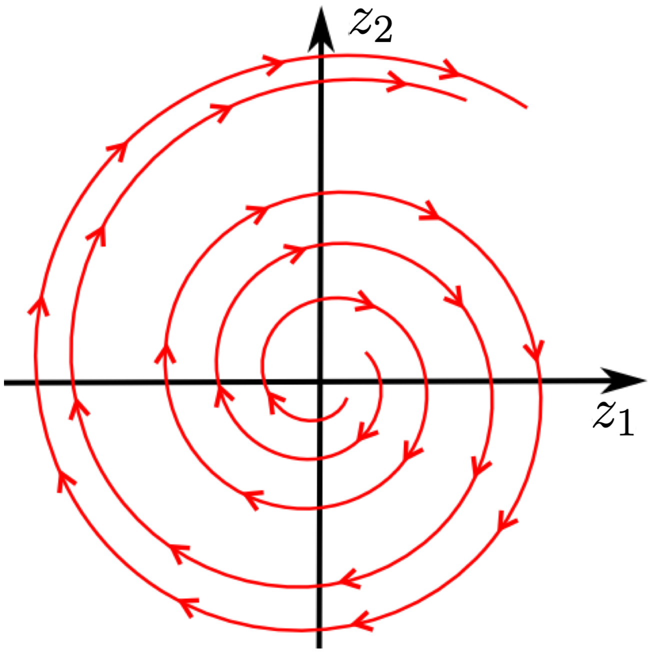

Figure 5: Phase Portrait

(g) Previously, it was shown that the trapezoidal region displayed in Figure 1 is forward invariant. In addition, the unique fixed point always lies inside the region. Thus, if it is an unstable node or spiral, applying the Poincaré-Bendixson Theorem, we can conclude that there exists an attracting limit cycle inside the trapezoidal region. Otherwise, we cannot say anything regarding the existence (or not) of a limit cycle. Thus, in case i) where $\sigma = 5$ and $\gamma = 6$ we cannot say anything while in case ii) where $\sigma = \frac{1}{2}$ and $\gamma = 3$ we can conclude there exists an attracting limit cycle inside the trapezoidal region.