Imperial College London

25 October 2022

Department of Bioengineering

Modelling in Biology (MiB), Prof Guy-Bart Stan & Dr Tom Ouldrdige

Warm-up exercises (to be done as homework)

The exercises that follow are meant to help you refresh your applied knowledge of simple mathematical concepts used throughout the course. Questions 3.3 onwards largely apply to the stochastic processes and networks component taught in the spring term.

Exercise 1: Differentiating and drawing functions

Consider the following functions:

- $f(x) = -x^2$
- $g(x) = \frac{1}{x}$
- $h(x) = \frac{1}{1+x^2}$
- $k(x) = \frac{x^2}{1+x^2}$

1. Differentiate these functions with respect to x analytically, i.e., write down the analytical expressions of $f'(x)$, $g'(x)$, $h'(x)$, and $k'(x)$.
2. Draw these functions by considering the asymptotic behaviours for $x \to \infty$, $x \to -\infty$, the zero-crossing points (values of x at which these functions are zero), the values at which the derivatives are zero (which correspond to extrema of the function), and the values at which the second derivatives are zero (which correspond to points of inflexion).

Exercise 2: Calculating the roots of quadratic equations

Calculate by hand the (complex) roots of these quadratic equations:

- $w^2 + 1 = 0$
- $x^2 + x + 1 = 0$
- $5y^2 + 2y + 3 = 0$
- $-2z^2 + 3z - 1 = 0$

Exercise 3: Scalars, vectors, matrices and eigenvalues

Consider the vectors: $a = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$ and $c = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, and the matrices: $X = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$ and $Y = \begin{pmatrix} 3 & 7 \\ 4 & 5 \end{pmatrix}$

1. Calculate by hand the following expressions:
    - $a^T b$
    - XY
    - Xc
    - $c^T Y$

    where ^{T} denotes the transposition operator.

2. Calculate by hand the eigenvalues and normalised eigenvectors of the following matrices:
    - $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
    - $B = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$
    - $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}$

3. One of the eigenvectors of the matrix below has an eigenvalue of 1. Find it.
    $M = \begin{pmatrix} 0.7 & 0 & 1 \\ 0.3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$

Exercise 4: Basic combinatorics

Show your calculation to the following two questions:

- Consider an urn with 6 balls numbered $1 \dots 6$. How many 2-element sets of balls can be drawn out of the urn if balls are not placed back in the urn after each draw?
- Consider an urn with 6 balls numbered $1 \dots 6$. How many 2-element sets of balls can be drawn out of the urn if balls can be placed back in the urn immediately after each draw?

Exercise 5: Integration

- Calculate the following definite integral: $\int_0^t \exp(-\nu\theta)\mathrm{d}\theta$

Exercise 6: Partial differentiation

- Calculate the following derivatives of $p(x,t) = \frac{1}{t}\exp(-2x^2 + t)$:
    1. $\frac{\partial p}{\partial t}$,
    2. $\frac{\partial p}{\partial x}$,
    3. $\frac{\partial^2 p}{\partial x^2}$.

Solutions

Question 1
- $f'(x) = -2x$, $g'(x) = -\frac{1}{x^2}$, $h'(x) = -\frac{2x}{(1+x^2)^2}$, $k'(x) = \frac{2x}{(1+x^2)^2}$

Question 2
- $w = \pm i$, $x = \frac{-1\pm i\sqrt{3}}{2}$, $y = \frac{-2\pm i\sqrt{56}}{10}$, $z = \{1, 0.5\}$

Question 3.1
- $a^T b = 9$, $XY = \begin{pmatrix} 3x_{1,1} + 4x_{1,2} & 7x_{1,1} + 5x_{1,2} \\ 3x_{2,1} + 4x_{2,2} & 7x_{2,1} + 5x_{2,2} \end{pmatrix}$, $Xc = \begin{pmatrix} \alpha x_{1,1} + \beta x_{1,2} \\ \alpha x_{2,1} + \beta x_{2,2} \end{pmatrix}$, $c^T Y = \begin{pmatrix} 3\alpha + 4\beta & 7\alpha + 5\beta \end{pmatrix}$

Question 3.2

matrix	eigenvalues	normalised eigenvectors
$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	$\lambda_1 = 1$, $\lambda_2 = 3$	$v_1 = \frac{\sqrt{2}}{2}\begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $v_2 = \frac{\sqrt{2}}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$
$B = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$	$\lambda_1 = 2$, $\lambda_2 = 2$	$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}$	$\lambda_1 = 2i$, $\lambda_2 = -2i$, $\lambda_3 = 1$	$v_1 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 \\ -1 \\ -i \end{pmatrix}$, $v_2 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 \\ -1 \\ i \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Question 3.3 $V = \begin{pmatrix} 10 \\ 3 \\ 3 \end{pmatrix}$ is an eigenvector of eigenvalue 1.

Question 4
- $\frac{n!}{k!(n-k)!}$ with $n = 6$ and $k = 2$ gives a total of 15 combinations.
- Adding the 6 pairs of repeated balls, the total is 21.

Question 5 $\int_0^t \exp(-\nu\theta)\mathrm{d}\theta = \frac{1}{\nu}\left(1 - \exp(-\nu t)\right)$.

Question 6
- $\frac{\partial p}{\partial t} = \frac{-1}{t^2}\exp(-2x^2 + t) + \frac{1}{t}\exp(-2x^2 + t) = (1 - 1/t)\, p(x,t)$,
- $\frac{\partial p}{\partial x} = \frac{-4x}{t}\exp(-2x^2 + t) = -4xp(x,t)$,
- $\frac{\partial^2 p}{\partial x^2} = \frac{-4}{t}\exp(-2x^2 + t) + \frac{16x^2}{t}\exp(-2x^2 + t) = 4\left(4x^2 - 1\right) p(x,t)$.