Toggle Switch

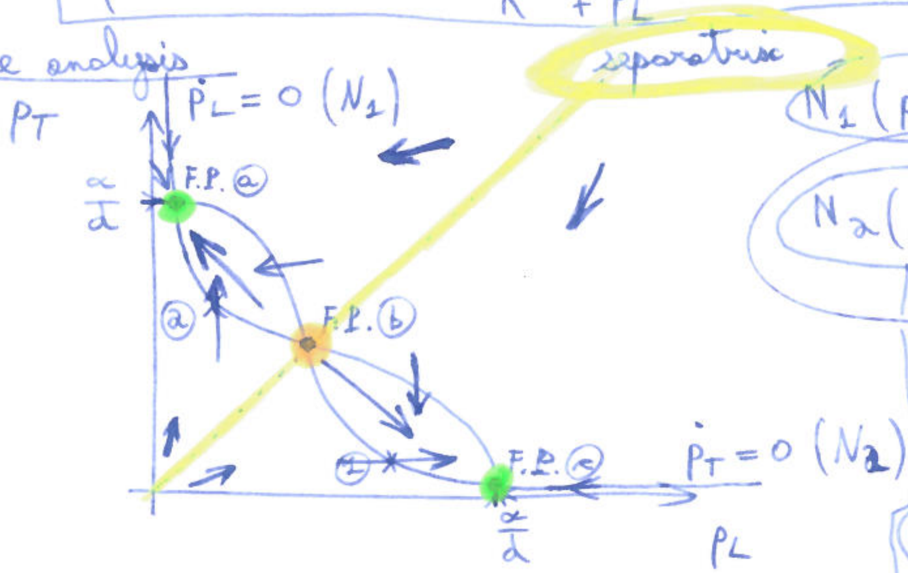
$$\begin{cases} \dot{p}_L = \alpha_L \dfrac{K_T^{n_T}}{K_T^{n_T} + p_T^{n_T}} - d_{L_2}\, p_L \\[2ex] \dot{p}_T = \alpha_T \dfrac{K_L^{n_L}}{K_L^{n_L} + p_L^{n_L}} - d_{T_2}\, p_T \end{cases}$$

Also, assume that the parameter values are chosen such that the toggle switch has 3 fixed points, e.g., n=2, alpha=10, K=1, d=1

Let's assume $\underbrace{\alpha_L = \alpha_T}_{=\alpha>0}$, $\underbrace{n_L = n_T}_{=n>0}$, $\underbrace{K_L = K_T}_{=K>0}$, $\underbrace{d_{L_2} = d_{T_2}}_{=d>0}$

$$\Rightarrow \boxed{\begin{cases} \dot{p}_L = \alpha \dfrac{K^n}{K^n + p_T^n} - d\, p_L \\[2ex] \dot{p}_T = \alpha \dfrac{K^n}{K^n + p_L^n} - d\, p_T \end{cases}}$$

Phase plane analysis



N_1 $(\dot{p}_L = 0)$: $p_L = \dfrac{\alpha}{d} \dfrac{K^n}{K^n + p_T^n}$

N_2 $(\dot{p}_T = 0)$: $p_T = \dfrac{\alpha}{d} \dfrac{K^n}{K^n + p_L^n}$

$$\frac{d}{\alpha} p_L = \frac{K^n}{K^n + p_T^n}$$

$$\Leftrightarrow K^n + p_T^n = \frac{K^n}{\frac{d}{\alpha} p_L}$$

$$(N_1) \Leftrightarrow \boxed{p_T = K \sqrt[n]{\frac{1 - \frac{d}{\alpha} p_L}{\frac{d}{\alpha} p_L}}}$$

On N_2 $(\dot{p}_T = 0)$, where do we have $\dot{p}_L > 0$?

$$\dot{p}_L > 0 \Leftrightarrow \alpha \frac{K^n}{K^n + p_T^n} - d\, p_L > 0$$

$$\Leftrightarrow \boxed{p_L < \frac{\alpha}{d} \frac{K^n}{K^n + p_T^n}} \quad N_1$$

$\Rightarrow$ For (1) on $\dot{p}_T = 0$, we have $\dot{p}_L > 0$

Similarly, On N_1 $(\dot{p}_L = 0)$, where do we have $\dot{p}_T > 0$?

$$\dot{p}_T > 0 \Leftrightarrow \alpha \frac{K^n}{K^n + p_L^n} - d\, p_T > 0$$

$$\Leftrightarrow \boxed{p_T < \frac{\alpha}{d} \frac{K^n}{K^n + p_L^n}} \quad N_2$$

$\Rightarrow$ For (a) on $\dot{p}_L = 0$, we have $\dot{p}_T > 0$

So the "middle" fixed point seems to be unstable.
It can be shown that this point is a saddle. The stable eigendirection of this saddle defines (at least locally around the saddle point) a separatrix in the phase plane. This separatrix delimitates the basins of attraction of the other 2 stable fixed points.

Local stability analysis

$$\begin{pmatrix} \dot{p}_L \\ \dot{p}_T \end{pmatrix} = \underbrace{\begin{pmatrix} -d & \alpha \overbrace{\operatorname{lin}\left(\frac{K^n}{K^n + p_T^n}\right)\Big|_{F.P.}}^{=\operatorname{lin}_1} \\ \alpha \overbrace{\operatorname{lin}\left(\frac{K^n}{K^n + p_L^n}\right)\Big|_{F.P.}}^{=\operatorname{lin}_2} & -d \end{pmatrix}}_{=J} \begin{pmatrix} p_L \\ p_T \end{pmatrix}$$

Eigenvalues of J

$$(\lambda + d)^2 - \alpha^2 \operatorname{lin}_1 \operatorname{lin}_2 = 0$$

$$\Leftrightarrow \boxed{\lambda_\pm = -d \pm \alpha \sqrt{\operatorname{lin}_1 \cdot \operatorname{lin}_2}}$$

- Slope of N_2 (i.e. $\dot{p}_T = 0$) : $\frac{\alpha}{d} \operatorname{lin}\left(\frac{K^n}{K^n + p_L^n}\right) = \frac{\alpha}{d} \operatorname{lin}_2 = s_2$

$$\boxed{N_2 : p_T = \frac{\alpha}{d} \frac{K^n}{K^n + p_L^n}}$$

- Slope of N_1 (i.e. $\dot{p}_L = 0$) → ⚠ we want the slope for the p_T vs p_L plot

$$\boxed{N_1 : p_L = \frac{\alpha}{d} \frac{K^n}{K^n + p_T^n}}$$

→ Let's linearise N_1

$$p_L = \frac{\alpha}{d} \underbrace{\operatorname{lin}\left(\frac{K^n}{K^n + p_T^n}\right)}_{=\operatorname{lin}_1} p_T$$

$$\Leftrightarrow p_T = \frac{d}{\alpha} \cdot \frac{1}{\operatorname{lin}_1} p_L$$

$$\Rightarrow \text{slope of } N_1 : \frac{d}{\alpha} \cdot \frac{1}{\operatorname{lin}_1} = s_1$$

Now: • at F.P. ⓑ : $\Delta_2 < \Delta_1 < 0$

$\alpha > 0$, $d > 0$
$lin_1 < 0$
$lin_2 < 0$

$$\Leftrightarrow \quad \frac{\alpha}{d}\, lin_2 < \frac{d}{\alpha}\,\frac{1}{lin_1} < 0$$

$$\Leftrightarrow \quad lin_1\, lin_2 > \frac{d^2}{\alpha^2} > 0 \quad \left(\text{since } lin_1 < 0 \text{ and } lin_2 < 0\right)$$

$$\Rightarrow \quad \bullet \sqrt{lin_1\, lin_2} > \frac{d}{\alpha} > 0$$

or

$$\bullet \; -\underbrace{\sqrt{lin_1\, lin_2}}_{>0} < -\frac{d}{\alpha} < 0$$

$$\Leftrightarrow \quad \bullet \boxed{\lambda_+ = -d + \alpha\sqrt{lin_1\, lin_2} > 0}$$

$$\bullet \boxed{\lambda_- = -d - \alpha\sqrt{lin_1\, lin_2} < 0}$$

$$\left(\text{since } \alpha > 0,\ d > 0,\ lin_1\, lin_2 > 0\right)$$

$\Rightarrow$ F.P. ⓑ is a <u>saddle point</u>.

• at F.P. ⓐ or at F.P. ⓒ: $0 > \Delta_2 > \Delta_1$

$$\Leftrightarrow \quad lin_1\, lin_2 < \frac{d^2}{\alpha^2}$$

$$\Rightarrow \quad \sqrt{lin_1\, lin_2} < \frac{d}{\alpha}$$

$$\Rightarrow \quad \boxed{\lambda_+ = -d + \alpha\sqrt{lin_1\, lin_2} < 0}$$

$$\left(\text{and again } \lambda_- = -d - \alpha\sqrt{lin_1\, lin_2} < 0 \text{ since } d > 0,\ \alpha > 0,\ lin_1\, lin_2 > 0\right)$$

$\Rightarrow$ F.P. ⓐ and F.P. ⓒ are both <u>stable nodes</u>