Engineering Tripos Part IIB/EIST Part II FOURTH YEAR

Module 4F2: Robust Multivariable Control

Solution to Examples Paper 4F2/2

1. (a) Clearly we need $a \geq 0$ (set $x_2 = 0$) and $c \geq 0$ (set $x_1 = 0$). Notice that

$$ax_1^2 + 2bx_1x_2 + cx_2^2 = \left(\sqrt{a}x_1 + \frac{b}{\sqrt{a}}x_2\right)^2 + \left(c - \frac{b^2}{a}\right)x_2^2$$

Therefore, we also need $c - \frac{b^2}{a} \geq 0$ (set $\sqrt{a}x_1 + \frac{b}{\sqrt{a}}x_2 = 0$), i.e. $b^2 \leq ac$, since $a \geq 0$. These conditions are also sufficient (from the above expression). Notice that $a \geq 0$ and $b^2 \leq ac$ in fact imply that $c \geq 0$.

(b) By definition

$$\begin{aligned} \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} > 0 \quad &\Leftrightarrow \quad \begin{bmatrix} x^T & y^T \end{bmatrix} \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} > 0 \\ &\Leftrightarrow \quad x^TAx + y^TCy + x^TBy + y^TB^Tx > 0 \end{aligned}$$

for all $\begin{bmatrix} x \\ y \end{bmatrix} \neq 0$. Clearly we need $A > 0$ (set $y = 0$ and choose appropriate x). This implies that A^{-1} exists. Therefore, the above inequality can be written as

$$\left(x + A^{-1}By\right)^T A \left(x + A^{-1}By\right) + y^T(C - B^TA^{-1}B)y > 0$$

Since $A > 0$ this is true for all y if and only if $C - B^TA^{-1}B > 0$ (otherwise, pick y to make the last term ≤ 0 and set $x = -A^{-1}By$).

The other case is symmetric.

2. (a)

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt} + \ldots + \frac{\partial V}{\partial x_n}\frac{dx_n}{dt} + \frac{\partial V}{\partial t}\frac{dt}{dt}$$

which is the required expression (in an expanded form).

(b)

$$
\begin{aligned}
\frac{\partial V}{\partial x_1} &= \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix} Xx + x^T X \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= 2x^T X \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{since } X^T = X \text{ and the above quantities are scalar} \\
\Rightarrow \frac{\partial V}{\partial x} &= \begin{bmatrix} 2x^T X \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & 2x^T X \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} & \ldots & 2x^T X \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \end{bmatrix} \\
&= 2x^T X
\end{aligned}
$$

(c) At $t = T$

$$
\begin{aligned}
& \min_u \left\{ x^T Q x + u^T R u + \frac{\partial V}{\partial x}(Ax + Bu) \right\} \\
= \; & \min_u \left\{ \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} Q + X_T A + A^T X_T & X_T B \\ B^T X_T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\} \\
= \; & x^T \left(Q + X_T A + A^T X_T - X_T B R^{-1} B^T X_T \right) x
\end{aligned}
$$

(using the lemma on quadratic forms in Handout 1).

This is a quadratic form in x. It therefore suggests the solution

$$
V(x,t) = x^T X(t) x \Rightarrow \frac{\partial V}{\partial t} = x^T \frac{dX}{dt} x
$$

where X solves

$$
-\frac{dX}{dt} = Q + XA + A^T X - XBR^{-1}B^T X
$$

(notice that this formula holds for $t = T$, therefore the above argument extends to all t).

3. $\frac{\partial V}{\partial t} = 0$ since the problem is time invariant: the minimum time it takes to bring the mass to rest does not depend on when you start the process. $c = 1$ since this gives

$$
J = \int_0^T c dt = \int_0^T dt = T
$$

Hence, minimising J minimises the total time it takes for the mass to come to rest, as required. The HJB equation then becomes

$$
\begin{aligned}
& \min_u \left(\frac{\partial V}{\partial x}\dot{x} + 1 \right) = 0 \\
\Rightarrow\ & \min_u \left(\frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + 1 \right) = 0 \\
\Rightarrow\ & \min_u \left(\frac{\partial V}{\partial x_1}x_2 + \frac{\partial V}{\partial x_2}u + 1 \right) = 0
\end{aligned}
$$

as required.

The optimal choice of u is $u = +1$ if $\frac{\partial V}{\partial x_2} < 0$ and $u = -1$ if $\frac{\partial V}{\partial x_2} > 0$. Any u is optimal if $\frac{\partial V}{\partial x_2} = 0$. Substituting back into the partial differential equation leads to

$$
\frac{\partial V}{\partial x_1}x_2 - \left| \frac{\partial V}{\partial x_2} \right| + 1 = 0
$$

By taking partial derivatives it is easy to show that the functions $V(x) = x_2 \pm \sqrt{2x_2^2 + 4x_1}$ solve the HJB equation when $u = -1$. By the above discussion, $u = -1$ implies that $\frac{\partial V}{\partial x_2} > 0$, which is a reasonable assumption based on the physics of the problem. Notice that $V > 0$ is also required (since it makes no sense to say that the mass stops in negative time).

4. (a) We have $Q = \alpha^2$, $R = 1$, $X_T = 1$, $A = 1$, $B = 1$. Hence $V(x,t) = x^T X(t) x = x^2 X(t)$, where X solves

$$
-\dot{X} = \alpha^2 + 2X - X^2, \quad X(T) = 1
$$

Taking derivatives of the suggested $X(t)$ it can be verified that it satisfies this equation. Therefore, the optimal cost is

$$
x_0^2 \left(1 + \sqrt{1+\alpha^2} \tanh\left(\sqrt{1+\alpha^2} T \right) \right)
$$

and the optimal control is

$$
u(t) = -X(t)x(t)
$$

(b)

$$
\dot{V} = 2x\dot{x}X = 2x(x+u)X
$$

Hence

$$
\begin{aligned}
\dot{V} + \alpha^2 x^2 + u^2 &= 2x^2 X + 2xuX + \alpha^2 x^2 + u^2 \\
&= (u + xX)^2 + (\alpha^2 + 2X - X^2)x^2
\end{aligned}
$$

Integrating from 0 to ∞ leads to

$$\lim_{t\to\infty} V(t) - V(0) + J = \|u + Xx\|_2^2 + (\alpha^2 + 2X - X^2)\|x\|_2^2$$

where $J = \int_0^\infty (\alpha^2 x^2 + u^2)dt$ (recall that $(\alpha^2 + 2X - X^2)$ is a scalar). If X is the stabilising solution to $(\alpha^2 + 2X - X^2) = 0$ and if u is the corresponding stabilising control, $x(t) \to 0$ as $t \to \infty$, therefore, $\lim_{t\to\infty} V(t) = 0$, and

$$J = V(0) + \|u + Xx\|_2^2$$

which is minimised at $u = -Xx$.

The solutions to $(\alpha^2 + 2X - X^2) = 0$ are $X = 1 \pm \sqrt{1+\alpha^2}$. They result in the closed loop system

$$\dot{x} = x + u = x - Xx = \mp\sqrt{1+\alpha^2}x$$

Therefore, the stabilising solution is $X = 1 + \sqrt{1+\alpha^2}$ and the optimal cost is $(1 + \sqrt{1+\alpha^2})x(0)^2$.

(c) Indeed,

$$\lim_{t\to-\infty} X(t) = 1 + \sqrt{1+\alpha^2}$$

5. Let $Z = XY$ where

$$X = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1m} \\ x_{21} & x_{22} & \dots & x_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nm} \end{bmatrix} \quad Y = \begin{bmatrix} y_{11} & y_{12} & \dots & y_{1n} \\ y_{21} & y_{22} & \dots & y_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ y_{m1} & y_{m2} & \dots & y_{mn} \end{bmatrix} \quad Z = \begin{bmatrix} z_{11} & z_{12} & \dots & z_{1n} \\ z_{21} & z_{22} & \dots & z_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ z_{n1} & z_{n2} & \dots & z_{nn} \end{bmatrix}$$

Then $z_{ii} = x_{i1}y_{1i} + x_{i2}y_{2i} + \ldots = \sum_{j=1}^m x_{ij}y_{ji}$. Therefore,

$$\begin{aligned} \text{trace}(Z) &= z_{11} + z_{22} + \ldots \\ &= x_{11}y_{11} + x_{12}y_{21} + \ldots \\ &\quad + x_{21}y_{12} + x_{22}y_{22} + \ldots \end{aligned}$$

The same expression is also obtained for trace(YX), by rearranging the order of the terms.

$$\begin{aligned} \text{trace}(B^T L_o B) &= \text{trace}(L_o B B^T) \\ &= -\text{trace}(L_o(L_c A^T + A L_c)) \\ &= -\text{trace}(L_c A^T L_o) - \text{trace}(L_c L_o A) \\ &= -\text{trace}(L_c(-C^T C)) \\ &= \text{trace}(C L_c C^T)) \end{aligned}$$

The claim about the 2-norm of G follows by the definitions (see also Handout 4).

6.

7. (a)

$$\begin{aligned} z_1 &= G_1 w_1 + G_2 z_2 \\ z_2 &= w_3 - u \\ y &= z_1 + W_1 w_2 \end{aligned}$$

In block matrix form

$$\begin{bmatrix} z_1 \\ z_2 \\ y \end{bmatrix} = \begin{bmatrix} G_1 & 0 & G_2 & -G_2 \\ 0 & 0 & I & -I \\ G_1 & W_1 & G_2 & -G_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ u \end{bmatrix}$$

(b)

$$\begin{aligned} z_1 &= G_1 w_1 + G_2 z_2 \\ z_2 &= w_3 - u \\ y &= z_1 + W_1 w_2 \\ e &= W_2 y - W_2 r \\ r &= r \end{aligned}$$

In block matrix form

$$\begin{bmatrix} z_1 \\ z_2 \\ e \\ y \\ r \end{bmatrix} = \begin{bmatrix} G_1 & 0 & G_2 & 0 & -G_2 \\ 0 & 0 & I & 0 & -I \\ W_2 G_1 & W_2 W_1 & W_2 G_2 & -W_2 & -W_2 G_2 \\ G_1 & W_1 & G_2 & 0 & -G_2 \\ 0 & 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ r \\ u \end{bmatrix}$$

8. From the block diagram,

$$\begin{aligned} \dot{x}_k &= A x_k + H(y - C x_k) + B(-F x_k) \\ &= (A - HC - BF) x_k + H y \\ u &= -F x_k \end{aligned}$$

For the closed loop system we have

$$\begin{aligned} \dot{x} &= A x - BF x_k \\ \dot{x}_k &= (A - HC - BF) x_k + HC x \end{aligned}$$

Let $e = x_k - x$. Then, $\dot{x} = (A - BF)x - BFe$ and

$$\dot{e} = \dot{x}_k - \dot{x} = (A - HC)(x_k - x) = (A - HC)e$$

i.e.

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BF & -BF \\ 0 & A - HC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

The eigenvalues of this matrix are the eigenvalues of $A - BF$ and the eigenvalues of $A - HC$. Hence the closed loop system is stable if and only if $A - BF$ and $A - HC$ are both stable.