Engineering Tripos Part IIB/EIST Part II FOURTH YEAR

**Module 4F2: Robust Multivariable Control**

**Examples Paper 4F2/2**

1. (a) Find necessary and sufficient conditions on $a$, $b$ and $c$ which guarantee that

   $$ax_1^2 + 2bx_1x_2 + cx_2^2 \geq 0 \text{ for all } x_1, x_2$$

   (b) If $A = A^T$ and $C = C^T$ show that

   $$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} > 0$$

   if, and only if, $A > 0$ and

   $$C - B^T A^{-1} B > 0$$

   Similarly, if $A = A^T$ and $C = C^T$ show that

   $$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} > 0$$

   if, and only if, $C > 0$ and

   $$A - BC^{-1}B^T > 0$$

   Hence, deduce that, if $A = A^T$, $C = C^T$, $A > 0$ and $C > 0$, then

   $$C - B^T A^{-1} B > 0 \iff A - BC^{-1}B^T > 0$$

2. *(this is the solution to the continuous time, finite horizon LQR problem which was omitted from the notes)*

   (a) Let $V(x, t)$ in $\mathbb{R}$ be a scalar valued function of a vector $x$ in $\mathbb{R}^n$ and $t$. Define, as usual,

   $$\frac{\partial V}{\partial x} = \left[\frac{\partial V}{\partial x_1}, \quad \frac{\partial V}{\partial x_2}, \quad \cdots, \quad \frac{\partial V}{\partial x_n}\right]$$

   and

   $$\frac{dx}{dt} = \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ \vdots \\ dx_n/dt \end{bmatrix}.$$

Show (using the standard formula for the total derivative of a function of several variables) that

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial t}$$

(b) Let $V = x^T X x$, where $X^T = X$. Show (by expanding) that

$$\frac{\partial V}{\partial x} = 2x^T X$$

(c) Consider the finite horizon, continuous time LQR problem (as defined in the lecture notes) with the associated HJB equation

$$-\frac{\partial V}{\partial t} = \min_u \left\{ x^T Q x + u^T R u + \frac{\partial V}{\partial x}(Ax + Bu) \right\}$$
$$V(x, T) = x^T X_T x$$

where $R > 0$ and assume $X_T = X_T^T$.

Show that, at $t = T$, the right hand side of the first expression above can be evaluated to give

$$-\left.\frac{\partial V}{\partial t}\right|_{t=T} = x^T(Q + X_T A + A^T X_T - X_T B R^{-1} B^T X_T)x$$

(Hint: use the result of (b), and note that $x^T X(Ax + Bu)$ is a scalar).

Hence, deduce that the solution to the HJB equation is of the form

$$V = x^T X(t) x$$

where $X(t)$ solves an appropriate Riccati differential equation.

3. Consider the problem of bringing a (unit) mass to rest in minimum time, using a force of bounded magnitude (also assumed to be equal to 1). This can be written as an optimal control problem, with state equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u$$

with $|u(t)| \leq 1$ for all $t$.

Show that the HJB equation for this problem can be written as

$$\min_{u:|u|\leq 1} \left( \frac{\partial V}{\partial x_1} x_2 + \frac{\partial V}{\partial x_2} u + 1 \right) = 0.$$

(You will need to explain why $\frac{\partial V}{\partial t} = 0$ and why the incremental cost $c = 1$).

Show that the optimal input $u$ is only ever equal to $\pm 1$, and find an expression for the optimal $u$ in terms of $V$.

Show that two solutions to the HJB equation are $V = x_2 \pm \sqrt{2x_2^2 + 4x_1}$. Over which region of the state-space do either of these correspond to the value function for the solution to the minimum time optimal control problem?

4. (a) Consider the following finite horizon LQR problem:

$$\text{minimize} \int_0^T \left(\alpha^2 x(t)^2 + u(t)^2\right)\, dt + x(T)^2$$

for the single input, single state system,

$$\dot{x} = x + u, \quad x(0) \neq 0$$

Show that the value function for this problem is given by

$$V(x,t) = x^2 X(t) \text{ where } X(t) = 1 - \sqrt{1+\alpha^2}\tanh\left(\sqrt{1+\alpha^2}(t-T)\right).$$

What are the optimal cost and the optimal control?

(b) Consider now the infinite horizon problem:

$$\text{minimize} \int_0^\infty \left(\alpha^2 x(t)^2 + u(t)^2\right)\, dt.$$

Take the value function

$$V(t) = Xx(t)^2$$

and show that

$$\dot{V}(t) + \alpha^2 x(t)^2 + u(t)^2 = (u + Xx)^2 + (2X + \alpha^2 - X^2)x^2.$$

Deduce that the optimal control is $u = -Xx$ where $X$ is the stabilizing solution to $2X + \alpha^2 - X^2 = 0$. What is the optimal cost?

(c) Find $\lim_{t\to-\infty} X(t)$, (where $X(t)$ is from your answer to part (a)) and verify that this corresponds to the stabilizing solution to the algebraic Riccati equation that you found in part (b).

5. Take two matrices $X$ and $Y$ such that $XY$ is square. Show that $\text{trace}(XY) = \text{trace}(YX)$. Hence prove that

$$\text{trace}(B^* L_o B) = \text{trace}(C L_c C^*)$$

where $L_o$ and $L_c$ are the controllability and observability Gramians.

Using this result, show from the state-space formulae for the $\mathcal{H}_2$ norm that $\|G(s)\|_2 = \|G^T(s)\|_2$.

6. In Matlab, use the commands `axxbc` and `trace` to find the the $\mathcal{H}_2$ norm of the transfer function:

$$\hat{G}(s) = \begin{bmatrix} \frac{50}{s+20} & \frac{1}{(s+3)(s+4)} \\ \frac{-10}{s+20} & \frac{1}{(s+5)(s^2+s+1)} \end{bmatrix}.$$

and compare your answer with that obtained for 4F2/1 Q2.

Demonstrate that the Ricatti equation derived in lectures for calculating the $\mathcal{H}_\infty$ norm has a solution for $\gamma > \|G\|_\infty$ (use `are`).

7. (a) Consider the closed-loop system of Fig. 2, in which $z_1 = \begin{bmatrix} G_1 & G_2 \end{bmatrix} \begin{bmatrix} w_1 \\ z_2 \end{bmatrix}$.

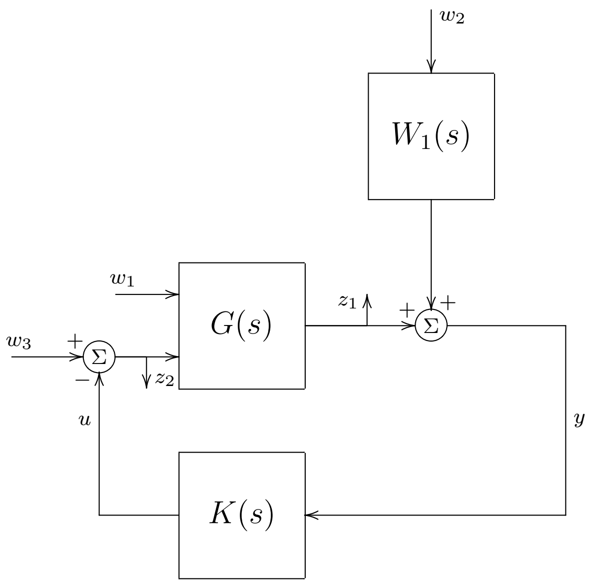



Fig. 2

Find a block transfer function matrix representation of the generalized plant $P(s)$ that has the property that

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathcal{F}_l\big(P(s), K(s)\big) \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

(Hint: $P$ itself should map $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ u \end{bmatrix}$ to $\begin{bmatrix} z_1 \\ z_2 \\ y \end{bmatrix}$, and should be independent of $K$)

(b) Now consider a modified system, where a reference input $r$ has been introduced. Let

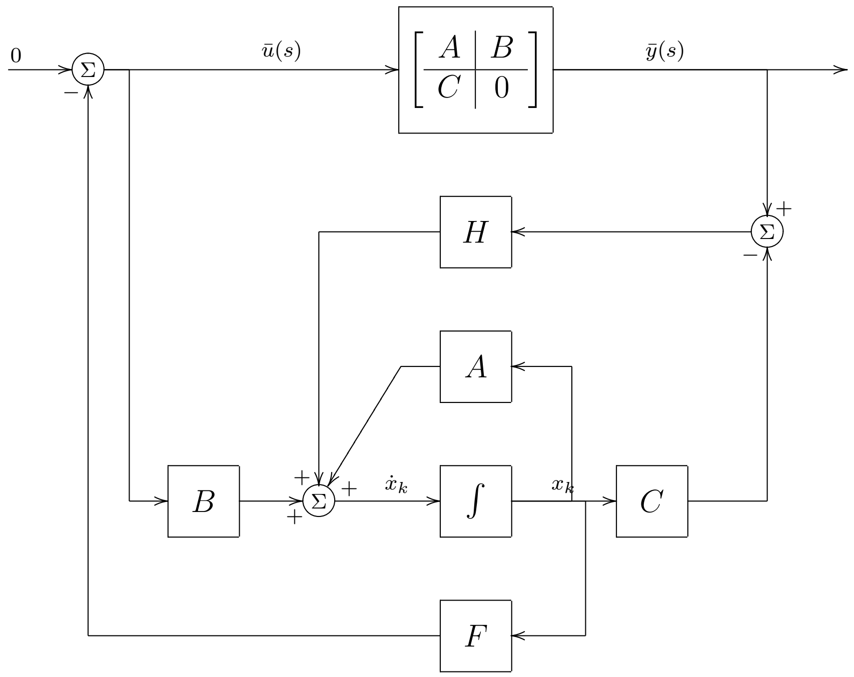
$$\bar{e}(s) = W_2(s)\big(\bar{y}(s) - \bar{r}(s)\big)$$

Find a block transfer function matrix representation of the generalized plant $P_2(s)$ which has the property that

$$\begin{bmatrix} z_1 \\ z_2 \\ e \end{bmatrix} = \mathcal{F}_l\big(P_2(s), K(s)\big) \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ r \end{bmatrix}.$$

(where now $u = K \begin{bmatrix} y \\ r \end{bmatrix}$)

8. Consider a controller in observer form as follows:




Show that the controller $K(s)$ (i.e. the transfer function for which $\bar{u}(s) = K(s)\bar{y}(s)$) has a state-space realization:

$$K(s) = \left[\begin{array}{c|c} A - BF - HC & H \\ \hline -F & 0 \end{array}\right]$$

By forming a state-space realization of the closed-loop system, with state vector

$$\left[\begin{array}{c} x \\ x_k - x \end{array}\right]$$

(where $x$ is the state vector of $G$), show that the closed-loop poles are given by the union of the eigenvalues of $A - BF$ and the eigenvalues of $A - HC$.

Answers:

3) $u = -\text{sgn}\frac{\partial V}{\partial x_2}$

4) (a) $\left(1 - \sqrt{1+\alpha^2}\tanh(-T\sqrt{1+\alpha^2})\right)x(0)^2$, $u(t) = -X(t)x(t)$.

(b) $X = 1 + \sqrt{1+\alpha^2}$, $(1 + \sqrt{1+\alpha^2})x(0)^2$

7) (a) $P = \begin{bmatrix} G_1 & 0 & G_2 & -G_2 \\ 0 & 0 & I & -I \\ G_1 & W_1 & G_2 & -G_2 \end{bmatrix}$

(b) $P = \begin{bmatrix} G_1 & 0 & G_2 & 0 & -G_2 \\ 0 & 0 & I & 0 & -I \\ W_2G_1 & W_2W_1 & W_2G_2 & -W_2 & -W_2G_2 \\ G_1 & W_1 & G_2 & 0 & -G_2 \\ 0 & 0 & 0 & I & 0 \end{bmatrix}$